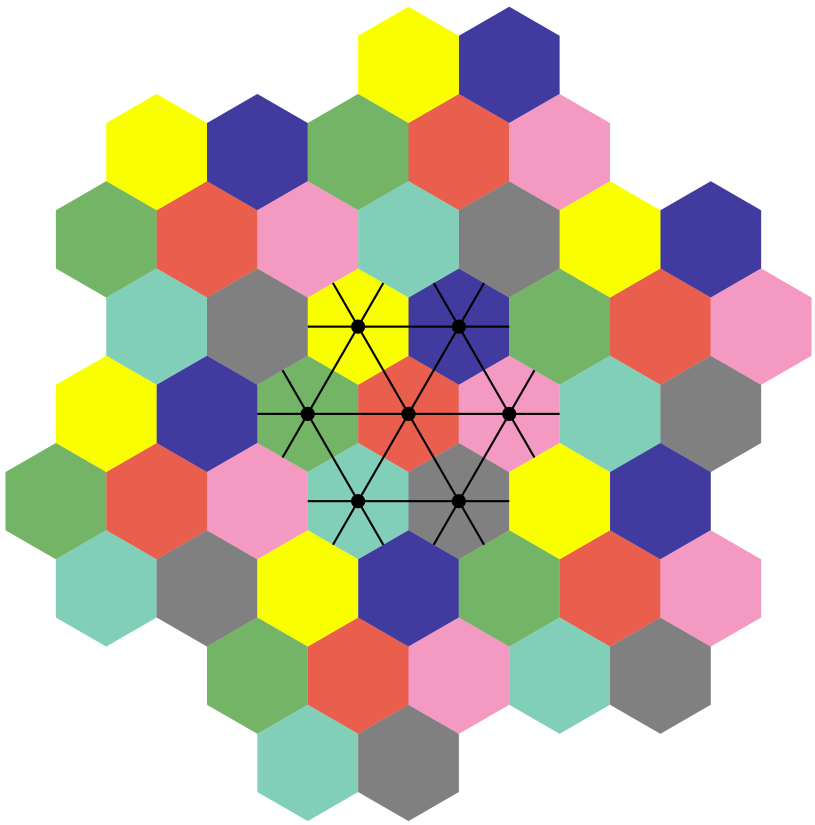
MATH 239/249

Introduction to Combinatorics



UNIVERSITY OF WATERLOO | FACULTY OF MATHEMATICS

Hello World, and Thanks!

Department of Combinatorics and Optimization
Faculty of Mathematics
University of Waterloo

Contents

Preliminaries.

MATH 239 is an introduction to two of the main areas in combinatorics – enumeration and graph theory. MATH 249 is an advanced version of MATH 239 intended for very strong students. These courses are designed for students in the second year of an undergraduate program in mathematics or computer science.

The prerequisites required from first-year mathematics are as follows.

- From MATH 135. Abstract algebra I: sets and propositional logic, proofs, mathematical induction, modular arithmetic, complex numbers, the Fundamental Theorem of Algebra.
- From MATH 136. Linear algebra I: systems of linear equations, Gaussian elimination, matrix algebra, vector spaces.
- From MATH 137. Calculus I: algebra with power series, open/closed sets, continuous functions, differentiation (but not integration).

We use the following standard notation for various number systems.

natural numbers	$\mathbb{N} = \{0, 1, 2, 3, ...\}$ including zero 0
integers	$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$
rational numbers	$\mathbb{Q}$
real numbers	$\mathbb{R}$
complex numbers	$\mathbb{C}$
integers (modulo n)	$\mathbb{Z}_n = \{[0], [1], [2], ..., [n-1]\}$
finite field of prime size	$\mathbb{F}_p = \mathbb{Z}_p$

The cardinality (size) of a set S is denoted by $|S|$.

For convenience, we use LHS and RHS as shorthand for "left-hand side" and "right-hand side", respectively.

Part I

Introduction to Enumeration

Overview.

Suppose I pay $5 for a lottery ticket – what is the chance that I win a share of the top prize? It depends on the details, of course. There are a certain number of ways to win, and a certain number of ways to lose. Enumeration is the art and science of figuring out this kind of thing. This is the subject of the first part of these course notes.

There are two broad principles of the subject. The combinatorial approach is to construct explicit one-to-one correspondences between sets to show that they have the same size. The algebraic approach is to translate the information about the problem from combinatorics into algebra, and then to use algebraic techniques to determine the sizes of the sets. We will see many examples of both approaches.

In Chapter 1 we begin by introducing the basic building blocks of the theory: subsets, lists and permutations, multisets, binomial coefficients, and so on. In Section 1.2 the use of these objects is illustrated by analyzing various applications and examples.

In Chapter 2 we introduce the idea of generating series. This begins with the Binomial Theorem and Binomial Series, which are of fundamental importance for later results. The general theory of generating series is developed in Section 2.2, and its use is illustrated by analyzing "compositions" in Section 2.3.

In Chapter 3 we consider the enumeration of various sets of binary strings, namely those which can be described by regular expressions – the "rational languages". This provides an interesting and varied class of examples to which the results of Chapters 2 and 4 apply.

In Chapter 4 we consider sequences which satisfy a homogeneous linear recurrence relation with initial conditions, the sequences arising in Chapters

2 and 3 being examples. This technique allows us to calculate the numbers which answer the various counting problems we have been considering. By using Partial Fractions we can derive an even better solution to such problems, although the calculations involved are also more complicated. (We include a proof of Partial Fractions for completeness.) In Section 4.4 we briefly discuss recurrence relations which are quadratic rather than linear.

Two additional topics are discussed in Chapters **??** and **??**.

Chapter 1

Basic Principles of Enumeration.

1.1 The Essential Ideas.

1.1.1 Choices – "AND" versus "OR".

In the next few pages we will often be constructing an object of some kind by repeatedly making a sequence of choices. In order to count the total number of objects we could construct we must know how many choices are available at each step, *but we must know more*: we also need to know how to combine these numbers correctly. A generally good guideline is to look for the words "AND" and "OR" in the description of the sequence of choices available. Here are a few simple examples.

Example 1.1. On a table before you are 7 apples, 8 oranges, and 5 bananas.

- *Choose an apple and a banana.*
  There are 7 choices for an apple AND 5 choices for a banana: $7 \times 5 = 35$ choices in all.
- *Choose an apple or an orange.*
  There are 7 choices for an apple OR 8 choices for an orange: $7 + 8 = 15$ choices in all.
- *Choose an apple and either an orange or a banana.*
  There are $7 \times (8 + 5) = 91$ possible choices.

- *Choose either an apple and an orange, or a banana.*
  There are $(7 \times 8) + 5 = 61$ possible choices.

Generally, "AND" corresponds to multiplication and "OR" corresponds to addition. The last two of the above examples show that it is important to determine exactly how the words "AND" and "OR" combine in the description of the problem.

From a mathematical point of view, "AND" corresponds to the Cartesian product of sets. If you choose one element of the set A AND you choose one element of the set B, then this is equivalent to choosing one element of the *Cartesian product* of A and B:

$$A \times B = \{(a, b) : \ a \in A \text{ and } b \in B\},$$

which is the set of all ordered pairs of elements (a, b) with $a \in A$ and $b \in B$. In general, the cardinalities of these sets are related by the formula

$$|A \times B| = |A| \cdot |B|.$$

Similarly, from a mathematical point of view, "OR" corresponds to the union of sets. If you choose one element of the set A OR you choose one element of the set B, then this is equivalent to choosing one element of the *union* of A and B:

$$A \cup B = \{c : \ c \in A \text{ or } c \in B\},$$

which is the set of all elements c which are either in A or in B.

It is not always true that $|A \cup B| = |A| + |B|$, because any elements in both A and B would be counted twice by $|A| + |B|$. The *intersection* of A and B is the set

$$A \cap B = \{c : \ c \in A \text{ and } c \in B\},$$

which is the set of all elements c which are both in A and in B. What is generally true is that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

(This is the first instance of the Principle of Inclusion/Exclusion, which will be discussed in general in Subsection 1.1.6.) In particular, if $A \cap B = \varnothing$ then $|A \cup B| = |A| + |B|$. Thus, in order to interpret "OR" as addition, it is important to check that the sets of choices A and B have no elements in common. Such a union of sets A and B for which $A \cap B = \varnothing$ is called a *disjoint union* of sets.

When solving enumeration problems it is usually very useful to describe a choice sequence for constructing the set of objects of interest, paying close attention to the words "AND" and "OR".

1.1.2 Lists, permutations, and subsets.

A *list* of a set S is a list of the elements of S exactly once each, in some order. For example, the lists of the set $\{1, a, X, g\}$ are:

$1aXg$	$a1Xg$	$X1ag$	$g1aX$
$1agX$	$a1gX$	$X1ga$	$g1Xa$
$1Xag$	$aX1g$	$Xa1g$	$ga1X$
$1Xga$	$aXg1$	$Xag1$	$gaX1$
$1gaX$	$ag1X$	$Xg1a$	$gX1a$
$1gXa$	$agX1$	$Xga1$	$gXa1$

A *permutation* is a list of the set $\{1, 2, ..., n\}$ for some $n \in \mathbb{N}$. A permutation $\sigma: \ a_1a_2...a_n$ can be interpreted as a function $\sigma: \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ by putting $\sigma(i) = a_i$ for all $1 \leq i \leq n$.

To construct a list of S we can choose any element v of S to be the first element in the list and follow this with any list of the set $S \smallsetminus \{v\}$. That is how the table above is arranged – each of the four columns corresponds to one choice of an element of $\{1, a, X, g\}$ to be the first element of the list. Within each column, all the lists of the remaining elements appear after the first element.

Let p_n denote the number of lists of an n-element set S. The first sentence of the previous paragraph is translated into the equation

$$p_n = n \cdot p_{n-1},$$

provided that n is positive. (In this equation there are n choices for the first element v of the list, AND p_{n-1} choices for the list of $S \smallsetminus \{v\}$ which follows it.) It is important to note here that each list of S will be produced exactly once by this construction.

Since it is easy to see that $p_1 = 1$ (and $p_2 = 2$), a simple proof by induction on n shows the following:

Theorem 1.2. *For every $n \geq 1$, the number of lists of an n-element set S is*

$$n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1.$$

In particular, taking $S = \{1, 2, ..., n\}$, this is the number of permutations of size n. The term *n factorial* is used for the number $n(n-1)\cdots 3 \cdot 2 \cdot 1$, and it is denoted by $n!$ for convenience. We also define $0!$ to be the number of lists of the 0-element (empty) set $\varnothing$. Since we want the equation $p_n = n \cdot p_{n-1}$ to hold when $n = 1$, and since $p_1 = 1! = 1$, we conclude that $0! = p_0 = 1$ as well.

A *subset* of a set S is a collection of some (perhaps none or all) of the elements of S, at most once each and in no particular order.

To specify a particular subset A of S, one has to decide for each element v of S whether v is in A or v is not in A. Thus we have two choices – $v \in A$ OR $v \notin A$ – for each element v of S. If $S = \{v_1, v_2, ..., v_n\}$ has n elements then the total number of choices is 2^n since we have 2 choices for v_1 AND 2 choices for v_2 AND ... AND 2 choices for v_n.

Theorem 1.3. *For every $n \geq 0$, the number of subsets of an n-element set is 2^n.*

A *partial list* of a set S is a list of a subset of S. That is, it is a list of some (perhaps none or all) of the elements of S, at most once each and listed in some particular order. We are going to count partial lists of length k of an n-element set.

First think about the particular case $n = 6$ and $k = 3$, and the set $S = \{a, b, c, d, e, f\}$. A partial list of S of length 3 is a list xyz of elements of S, which must all be different. There are:
6 choices for x (since x is in S), AND
5 choices for y (since $y \in S$ but $y \neq x$), AND
4 choices for z (since $z \in S$ but $z \neq x$ and $z \neq y$).
Altogether there are $6 \cdot 5 \cdot 4 = 120$ partial lists of $\{a, b, c, d, e, f\}$ of length 3.

This kind of reasoning works just as well in the general case. If S is an n-element set and we want to construct a partial list $v_1v_2...v_k$ of elements of S of length k, then there are:
n choices for v_1, AND
$n-1$ choices for v_2, AND
....
$n-(k-2)$ choices for v_{k-1}, AND
$n-(k-1)$ choices for v_k.

This proves the following result.

Theorem 1.4. *For $n, k \geq 0$, the number of partial lists of length k of an n-element set is $n(n-1)\cdots(n-k+2)(n-k+1)$.*

Notice that if $k > n$ then the number 0 will appear as one of the factors in the product $n(n-1)\cdots(n-k+2)(n-k+1)$. This makes sense, because if $k > n$ then there are no partial lists of length k of an n-element set. On the other hand, if $0 \leq k \leq n$ then we could also write this product as

$$n(n-1)\cdots(n-k+2)(n-k+1) = \frac{n!}{(n-k)!}.$$

We next count subsets of an n-element set S which have a particular size k. So for $n, k \geq 0$ let $\binom{n}{k}$ denote the number of k-element subsets of an n-element set S. Notice that if $k < 0$ or $k > n$ then $\binom{n}{k} = 0$ because in these cases it is impossible for S to have a k-element subset. Thus we need only consider k in the range $0 \leq k \leq n$.

To count k-element subsets of S we consider another way of constructing a partial list of length k of S. Specifically, we can choose a k-element subset A of S AND a list of A. The result will be a list of a subset of S of length k. Since every partial list of length k of S is constructed exactly once in this way, this translates into the equation

$$\binom{n}{k} \cdot k! = \frac{n!}{(n-k)!}.$$

In summary, we have proved the following result.

Theorem 1.5. *For $0 \le k \le n$, the number of k-element subsets of an n-element set is*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The numbers $\binom{n}{k}$ are read as "n choose k" and are called *binomial coefficients*.

1.1.3 Think of what the numbers mean.

Usually, when faced with a formula to prove, one's first thought is to prove it by algebraic calculations, or perhaps with an induction argument, or maybe with a combination of the two. But often that is not the easiest way, nor is it the most informative. A much better strategy is one which gives some insight into the meaning of all of the parts of the formula. If we can interpret all the numbers as counting things, addition as "OR", and multiplication as "AND", then we can hope to find an explanation of the formula by constructing some objects in the correct way.

Example 1.6. Consider the equation, for any $n \ge 0$:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

This could be proved by induction on n, but many more details would have to be given and the proof would not address the true "meaning" of the formula. Instead, let's interpret everything combinatorially:

- the number of subsets of the n-element set $\{1, 2, ..., n\}$ is 2^n;
- for each $0 \le k \le n$, the number of k-element subsets of $\{1, 2, ..., n\}$ is $\binom{n}{k}$;
- addition corresponds to "OR" (that is, disjoint union of sets).

So, this formula is saying that choosing a subset of $\{1, 2, ..., n\}$ (in one of 2^n ways) is equivalent to choosing a k-element subset of $\{1, 2, ..., n\}$ (in one of $\binom{n}{k}$ ways) for exactly one value of k in the range $0 \le k \le n$. Said that way the formula becomes self–evident, and there is nothing more to prove.

Example 1.7. Consider the equation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

where we are using the fact that $\binom{m}{j} = 0$ if $j < 0$ or $j > m$.

This equation can be proven algebraically from the formula of Theorem 1.5, and that is a good exercise which I encourage you to try. But a more informative proof interprets these numbers combinatorially as follows:

- $\binom{n}{k}$ is the number of k-element subsets of $\{1, 2, ..., n\}$;
- $\binom{n-1}{k-1}$ is the number of $(k-1)$-element subsets of $\{1, 2, ..., n-1\}$;
- $\binom{n-1}{k}$ is the number of k-element subsets of $\{1, 2, ..., n-1\}$;
- addition corresponds to disjoint union of sets.

So, this equation is saying that choosing a k-element subset A of $\{1, 2, ..., n\}$ is equivalent to either choosing a $(k-1)$-element subset of $\{1, 2, ..., n-1\}$ or a k-element subset of $\{1, 2, ..., n-1\}$. This is perhaps not as clear as the previous example, but the two cases depend upon whether the chosen k-element subset A of $\{1, 2, ..., n\}$ is such that $n \in A$ OR $n \notin A$. If $n \in A$ then $A \smallsetminus \{n\}$ is a $(k-1)$-element subset of $\{1, 2, ..., n-1\}$, while if $n \notin A$ then A is a k-element subset of $\{1, 2, ..., n-1\}$. This construction explains the correspondence, proving the formula.

This principle – interpreting equations combinatorially and proving the formulas by describing explicit correspondences between sets of objects – is one of the most important and powerful ideas in enumeration. We will apply this way of thinking throughout the first part of these notes.

Incidentally, the equation in Example 1.7 is a very useful recurrence relation for computing binomial coefficients quickly. Together with the facts

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$$

and $\binom{n}{0} = \binom{n}{n} = 1$ it can be used to compute any number of binomial coefficients without difficulty. The resulting table is known as *Pascal's Triangle*:

$n\backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

1.1.4 Multisets.

Imagine a bag which contains a large number of marbles of three colours – red, green, and blue, say. The marbles are all indistinguishable from one another except for their colours. There are N marbles of each colour, where N is very, very large (more precisely we should be considering the limit as $N \to \infty$). If I reach into the bag and pull out a handful of 11 marbles, I will have r red marbles, g green marbles, and b blue marbles, for some nonnegative integers (r, g, b) such that $r + g + b = 11$. How many possibile outcomes are there?

The word "multiset" is meant to suggest a set in which the objects can occur more than once. For example, the outcome $(4, 5, 2)$ in the above situation corresponds to the "set" $\{R, R, R, R, G, G, G, G, G, B, B\}$ in which R is a red marble, G is a green marble, and B is a blue marble. This is an 11-element multiset with elements of three types. The number of these multisets is the solution to the above problem.

Definition 1.8. Let $n \geq 0$ and $t \geq 1$ be integers. A *multiset of size n with elements of t types* is a sequence of nonnegative integers $(m_1, ..., m_t)$ such that

$$m_1 + m_2 + \cdots + m_t = n.$$

The interpretation is that m_i is the number of elements of the multiset which are of the i-th type, for each $1 \leq i \leq t$.

Theorem 1.9. *For any $n \geq 0$ and $t \geq 1$, the number of n-element multisets with elements of t types is*

$$\binom{n+t-1}{t-1}.$$

Proof. Think of what that number means! By Theorem 1.5, $\binom{n+t-1}{t-1}$ is the number of $(t-1)$-element subsets of an $(n+t-1)$-element set. So, let's write down a row of $(n+t-1)$ circles from left to right:

O O O O O O O O O O O O O

and cross out some $t-1$ of these circles to choose a $(t-1)$-element subset:

O O O O X O O O O O X O O

Now the $t-1$ crosses chop the remaining sequence of n circles into t segments of consecutive circles. (Some of these segments might be empty, which is to say of length zero.) Let m_i be the length of the i-th segment of consecutive O-s in this construction. Then $m_1 + m_2 + \cdots + m_t = n$, so that $(m_1, m_2, ..., m_t)$ is an n-element multiset with t types. Conversely, if $(m_1, m_2, ..., m_t)$ is an n-element multiset with t types then write down a sequence of m_1 O-s, then an X, then m_2 O-s, then an X, and so on, finishing with an X and then m_t O-s. The positions of the X-s will indicate a $(t-1)$-element subset of the positions $\{1, 2, ..., n+t-1\}$.

The construction of the above paragraph shows how to translate between $(t-1)$-element subsets of $\{1, 2, ..., n+t-1\}$ and n-element multisets with t types of element. This one–to–one correspondence completes the proof of the theorem. $\square$

To answer the original question of this section, the number of 11-element multisets with elements of 3 types is $\binom{11+3-1}{3-1} = \binom{13}{2} = 78$.

1.1.5 Bijective proofs.

The arguments above, counting lists, permutations, subsets, multisets, and so on, can be phrased more formally using the idea of bijections between finite sets. In simple cases as we have seen so far this is not always necessary, but it is good style. In more complicated situations, as we will see in Chapters 2 to 4, it is a very useful way to organize one's thoughts.

Definition 1.10. Let $f : \mathcal{A} \to \mathcal{B}$ be a function from a set $\mathcal{A}$ to a set $\mathcal{B}$.

- The function f is *surjective* if for every $b \in \mathcal{B}$ there exists an $a \in \mathcal{A}$ such that $f(a) = b$.
- The function f is *injective* if for every $a, a' \in \mathcal{A}$, if $f(a) = f(a')$, then $a = a'$.
- The function f is *bijective* if it is both surjective and injective.
- The notation $\mathcal{A} \rightleftharpoons \mathcal{B}$ indicates that there is a bijection between the sets $\mathcal{A}$ and $\mathcal{B}$.

Functions with these properties are called surjections, injections, or bijections, respectively. An older terminology – now out of fashion – is that surjections are "onto" functions, injections are "one-to-one" functions, and bijections are "one-to-one and onto". By Exercise 1.4(a), the relation $\rightleftharpoons$ is an equivalence relation.

The point of Definition 1.10 is the following. Consider a bijection $f : \mathcal{A} \to \mathcal{B}$. Then every $b \in \mathcal{B}$ is the image of at least one $a \in \mathcal{A}$, since f is surjective. On the other hand, every $b \in \mathcal{B}$ is the image of at most one $a \in \mathcal{A}$, since f is injective. Therefore, every $b \in \mathcal{B}$ is the image of exactly one $a \in \mathcal{A}$. In other words, the relation $f(a) = b$ pairs off all the elements of $\mathcal{A}$ with all the elements of $\mathcal{B}$. It follows that $\mathcal{A}$ and $\mathcal{B}$ have the same number of elements. That is, if $\mathcal{A} \rightleftharpoons \mathcal{B}$ then $|\mathcal{A}| = |\mathcal{B}|$. The converse implication holds, and for infinite sets the relation $\mathcal{A} \rightleftharpoons \mathcal{B}$ is taken as the *definition* of two sets "having the same size".

Proposition 1.11. *Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{A}$ be functions between two sets $\mathcal{A}$ and $\mathcal{B}$. Assume the following.*

- *For all $a \in \mathcal{A}$, $g(f(a)) = a$.*
- *For all $b \in \mathcal{B}$, $f(g(b)) = b$.*

Then both f and g are bijections. Moreover, for $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we have $f(a) = b$ if and only if $g(b) = a$.

Proof. Exercise 1.4(b). $\square$

A pair of functions as in Proposition 1.11 are called *mutually inverse bijections*. The notation $g = f^{-1}$ and $f = g^{-1}$ is used to denote this relation. Notice that for a bijection f, we have $(f^{-1})^{-1} = f$.

Here are two examples of this way of thinking.

Example 1.12 (Subsets and indicator vectors.)**.** Let $\mathcal{P}(n)$ be the set of all subsets of $\{1, 2, ..., n\}$, and let $\{0, 1\}^n$ be the set of all *indicator vectors* $\alpha = (a_1, a_2, ..., a_n)$ in which each coordinate is either 0 or 1. There is a bijection between these two sets. For a subset $S \subseteq \{1, 2, ..., n\}$, define the vector $\alpha(S) = (a_1(S), a_2(S), ..., a_n(S))$ by saying that for each $1 \leq i \leq n$,

$$a_i(S) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

Conversely, for an indicator vector $\alpha = (a_1, ..., a_n)$ define a subset $S(\alpha)$ by saying that

$$S(\alpha) = \{i \in \{1, 2, ..., n\} : \ a_i = 1\}.$$

For example, when $n = 8$ the subset $\{2, 3, 5, 7\}$ corresponds to the indicator vector $(0, 1, 1, 0, 1, 0, 1, 0)$. The constructions $S \mapsto \alpha(S)$ and $\alpha \mapsto S(\alpha)$ are mutually inverse bijections between the sets $\mathcal{P}(n)$ and $\{0, 1\}^n$ as in Proposition 1.11. It follows that $|\mathcal{P}(n)| = |\{0, 1\}^n| = 2^n$. This is a formalization of the proof of Theorem 1.3.

Example 1.13 (Subsets and multisets.)**.** The proof of Theorem 1.9 can be phrased in terms of bijections, as follows.

Let $\mathcal{M}(n, t)$ be the set of all multisets of size $n \in \mathbb{N}$ with elements of $t \geq 1$ types. Let $\mathcal{B}(a, k)$ be the set of all k-element subsets of $\{1, 2, ..., a\}$. We establish a bijection between $\mathcal{M}(n, t)$ and $\mathcal{B}(n+t-1, t-1)$ in what follows. Theorem 1.5 implies that $|\mathcal{B}(n + t - 1, t - 1)| = \binom{n+t-1}{t-1}$, completing the proof of Theorem 1.9. Here is a precise description of this bijection.

Let S be any $(t - 1)$-element subset of $\{1, 2, ..., n + t - 1\}$. We can sort the elements of S in increasing order: $S = \{s_1, s_2, ..., s_{t-1}\}$ in which $s_1 < s_2 < \cdots < s_{t-1}$. For notational convenience, let $s_0 = 0$ and let $s_t = n + t$. Now define a sequence $\mu = (m_1, m_2, ..., m_t)$ by letting $m_i = s_i - s_{i-1} - 1$ for all $1 \leq i \leq t$.

For example, with $n = 10$ and $t = 4$, consider the 3-element subset $S = \{2, 7, 11\}$ of $\{1, 2, ..., 13\}$. Then $s_0 < s_1 < \cdots < s_t$ is $0 < 2 < 7 < 11 < 14$, the sequence of differences is $(2, 5, 4, 3)$, and subtracting 1 from each of these yields $\mu = (1, 4, 3, 2)$. Notice that μ is a multiset of size 10 with 4 types of elements.

In general, since $s_{i-1} < s_i$ for all $1 \leq i \leq t$, it follows that $m_i = s_i - s_{i-1} - 1$ is a nonnegative integer. Also, since $s_t = n + t$, it follows that $m_1 + m_2 + \cdots + m_t = s_t - t = n$. That is, μ is a multiset of size n with elements of t types. This describes a function $S \mapsto \mu$ from the set $\mathcal{B}(n + t - 1, t - 1)$ to the set $\mathcal{M}(n, t)$. We claim that this function is a bijection between these two sets.

To show that our construction $S \mapsto \mu$ is a bijection, we will describe its inverse function. Begin with a multiset $\mu = (m_1, m_2, ..., m_t)$ of size n with t types of elements. For each $1 \leq i \leq t-1$, let $s_i = m_1 + m_2 + \cdots + m_i + i$. Notice that

$$1 \leq s_1 < s_2 < \cdots < s_{t-1} \leq n + t - 1.$$

Therefore, $S = \{s_1, s_2, ..., s_{t-1}\}$ is a member of the set $\mathcal{B}(n + t - 1, t - 1)$.

To finish this example, one must check that these constructions, $S \mapsto \mu$ and $\mu \mapsto S$, are mutually inverse bijections as in Proposition 1.11. The details are left as Exercise 1.6.

1.1.6 Inclusion/Exclusion.

In a vase is a bouquet of flowers. Each flower is (at least one of) fresh, fragrant, or colourful:
(a) 11 flowers are fresh;
(b) 7 flowers are fragrant;
(c) 8 flowers are colourful;
(d) 6 flowers are fresh and fragrant;
(e) 5 flowers are fresh and colourful;
(f) 2 flowers are fragrant and colourful;
(g) 2 flowers are fresh, fragrant, and colourful.
How many flowers are in the bouquet?

The Principle of Inclusion/Exclusion is a systematic method for answering such questions, which involve overlapping conditions which can be sat-

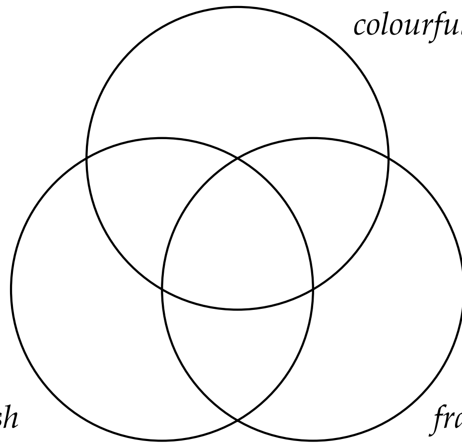


Figure 1.1: A Venn diagram for three sets.

isfied (or not) in various combinations.

Example 1.14. For a small problem as above we can reason backwards as follows:
(g): there are 2 flowers with all three properties (fresh, fragrant, and colourful);
(h): from (g) and (f) there are 0 flowers which are fragrant and colourful but not fresh;
(i): from (g) and (e) there are 3 flowers which are fresh and colourful but not fragrant;
(j): from (g) and (d) there are 4 flowers which are fresh and fragrant but not colourful;
(k): from (c)(g)(h)(i) there are 3 flowers which are colourful but neither fresh not fragrant;
(ℓ): from (b)(g)(h)(j) there is 1 flower which is fragrant but neither fresh nor colourful;
(m): from (a)(g)(i)(j) there are 2 flowers which are fresh but neither fragrant nor colourful.

The total number of flowers is counted by the disjoint union of the cases (g) through (m); that is $2 + 0 + 3 + 4 + 3 + 1 + 2 = 15$.

A Venn diagram is extremely useful for organizing this calculation. Figure 1.1 is a Venn diagram for the three sets involved in this question. Item (g) in

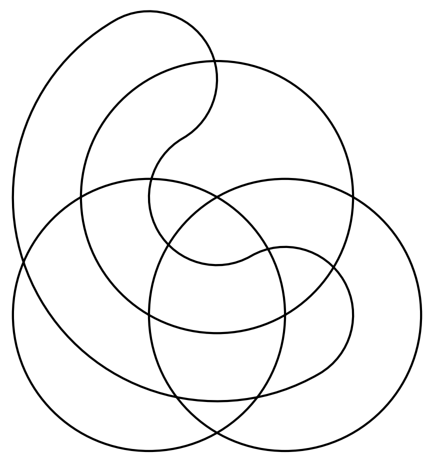


Figure 1.2: A Venn diagram for four sets.

the original data gives the number of flowers counted in the central triangle. The subsequent steps (h) to (m) calculate the rest of the numbers in the diagram, moving outwards from the center.

The above works very well for three properties (fresh, fragrant, colourful) but becomes increasingly difficult to apply as the number of properties increases. Figure 1.2 shows a Venn diagram for four sets, for instance. Instead, consider this alternative to the calculation in Example 1.14:

$$(a) + (b) + (c) - (d) - (e) - (f) + (g) = 11 + 7 + 8 - 6 - 5 - 2 + 2 = 15.$$

This looks much easier to apply, and it gives the right answer. Why? That is the Principle of Inclusion/Exclusion, which we now explain in general.

Let $A_1, A_2, ..., A_m$ be finite sets. We want a formula for the cardinality of the union of these sets $A_1 \cup A_2 \cup \cdots \cup A_m$. First a bit of notation: if S is a nonempty subset of $\{1, 2, ..., m\}$ then let A_S denote the intersection of the sets A_i for all $i \in S$. So, for example, with this notation we have $A_{\{2,3,5\}} = A_2 \cap A_3 \cap A_5$.

Theorem 1.15 (Inclusion/Exclusion)**.** *Let A_1, A_2,..., A_m be finite sets. Then*

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = \sum_{\varnothing \neq S \subseteq \{1,...,m\}} (-1)^{|S|-1} |A_S|.$$

We prove Theorem 1.15 in Section 2.5, but all that is required is the Binomial Theorem 2.2.

1.1.7 Combinatorial probabilities.

We can interpret counting problems in terms of probabilities by making one additional hypothesis. That hypothesis is that **every possible outcome is equally likely**. The exact definition of what is an "outcome" depends on the particular problem. If Ω denotes a finite set of all possible outcomes, then any subset E of Ω is what a probabilist calls an "event". The probability that a randomly chosen outcome from Ω is in the set E is $|E|/|\Omega|$ exactly because every outcome has probability $1/|\Omega|$ of being chosen, and there are $|E|$ elements in E. Here are a few examples to illustrate these ideas.

Example 1.16. What is the probability that a random subset of $\{1, 2, ..., 8\}$ has at most 3 elements?

Here an outcome is a subset of $\{1, 2, ..., 8\}$, and there are $2^8 = 256$ such subsets. The number of subsets of $\{1, 2, ..., 8\}$ with at most 3 elements is

$$\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} = 1 + 8 + 28 + 56 = 93.$$

So the probability in question is

$$\frac{93}{256} = 0.363281...$$

to six decimal places.

Example 1.17. What is the probability that a random list of $\{a, b, c, d, e, f\}$ contains the letters fad as a consecutive subsequence?

Here an outcome is a list of $\{a, b, c, d, e, f\}$, and there are $6! = 720$ such lists. Those lists of this set which contain fad as a consecutive subsequence can be constructed uniquely as the lists of the set $\{b, c, e, fad\}$, so there are $4! = 24$ of these. Thus, the probability in question is

$$\frac{24}{720} = \frac{1}{30} = 0.03333...$$

Example 1.18. What is the probability that a randomly chosen 2-element multiset with t types of element has both elements of the same type?

The outcomes are the 2-element multisets with t types, numbering

$$\binom{2+t-1}{t-1} = \binom{t+1}{t-1} = \binom{t+1}{2} = \frac{(t+1)t}{2}$$

in total. Of these, exactly t of them have both elements of the same type – choose one of the t types and take two elements of that type. Thus, the probability in question is

$$\frac{2t}{(t+1)t} = \frac{2}{t+1}.$$

The values for the first few t are given in the following table to four decimal places:

t	1	2	3	4	5	6	7
	1.0000	0.6667	0.5000	0.4000	0.3333	0.2857	0.2500

1.2 Examples and Applications.

1.2.1 The Vandermonde convolution formula.

Example 1.19 (Vandermonde convolution formula)**.** For $m, n, k \in \mathbb{N}$,

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}.$$

For instance, with $m = 4$ and $n = 2$ and $k = 3$ this says that

$$\binom{6}{3} = \binom{4}{0}\binom{2}{3} + \binom{4}{1}\binom{2}{2} + \binom{4}{2}\binom{2}{1} + \binom{4}{3}\binom{2}{0}.$$

(Of course, $\binom{2}{3} = 0$, but that doesn't matter.)

The Vandermonde convolution formula can be proven algebraically by induction on $m + n$, but the proof is finicky and doesn't give much insight into what the formula "means". (The formula can also be deduced easily from the Binomial Theorem **??**, as we will see in Example 2.3.)

Here is a direct **combinatorial** proof, illustrating the strategy of thinking about what the numbers mean. On the LHS, $\binom{m+n}{k}$ is the number of k-element subsets S of the set $\{1, 2, ..., m + n\}$. On the RHS, the number can be produced as follows:

- choose a value of j in the range $0 \le j \le k$, and
- choose a j-element subset A of $\{1, 2, ..., m\}$, and
- choose a $(k - j)$-element subset B of $\{m + 1, ..., m + n\}$.

(Notice that the set $\{m + 1, ..., m + n\}$ has n elements, so it has $\binom{n}{k-j}$ subsets of size $k - j$.) Now the formula is proved by describing a bijection between the k-element subsets S of $\{1, 2, ..., m+n\}$ counted on the LHS, and the pairs (A, B) of subsets counted on the RHS. To describe this correspondence, let $M = \{1, 2, ..., m\}$ and $N = \{m + 1, ..., m + n\}$. Notice that $M \cap N = \varnothing$ and $M \cup N = \{1, 2, ..., m+n\}$ and $|M| = m$ and $|N| = n$. Now, given a k-element subset S of $\{1, 2, ..., m + n\}$ we let

$$A = S \cap M \quad \text{and} \quad B = S \cap N.$$

Conversely, given a pair of subsets (A, B) satisfying the conditions in the points above, we let $S = A \cup B$. After some thought, you will see that these constructions $S \mapsto (S \cap M, S \cap N)$ and $(A, B) \mapsto A \cup B$ are mutually inverse bijections between the sets in question. Therefore, there are the same number of objects on each side, and the formula is proved.

1.2.2 Common birthdays.

Example 1.20. Let $p(n)$ denote the probability that in a randomly chosen group of n people, at least two of them are born on the same day of the year. What does the function $p(n)$ look like?

To simplify the analysis, we will ignore the existence of leap years and assume that every year has exactly 365 days. Moreover, we will also assume that people's birthdays are independently and uniformly distributed over the 365 days of the year, so that we can use the ideas of combinatorial probability theory. These are reasonable approximations – although they introduce tiny errors they do not change the qualitative "shape" of the answer.

To begin with, $p(1) = 0$ since there is only $n = 1$ person in the group. Also, if $n > 365$ then $p(n) = 1$ since there are more people in the group than days in a year, so at least two people in the group must have the same birthday.

For n in the range $2 \leq n \leq 365$ it is quite complicated to analyze the probability $p(n)$ directly. However, the complementary probability $1 - p(n)$ is relatively easy to compute. From the definition of $p(n)$ we see that $1-p(n)$ is the probability that in a randomly chosen group of n people, **no two of them** are born on the same day of the year. This model is equivalent to rolling "no pair" when throwing n independent dice each with 365 sides. If we list the people in the group as P_1, P_2, ..., P_n in any order, then their birthdays must form a partial list of the 365 days of the year, of length n. There are $365!/(365 - n)!$ such partial lists. Since the total number of outcomes is 365^n, we have derived the formula

$$1 - p(n) = \frac{365!}{(365 - n)!365^n}.$$

Therefore

$$p(n) = 1 - \frac{365!}{(365 - n)!365^n}.$$

To give some feeling for what this looks like, here is a table of $p(n)$ (rounded to six decimal places) for selected values of $2 \leq n \leq 365$.

n	$p(n)$	n	$p(n)$	n	$p(n)$
2	0.002740	25	0.568700	70	0.999160
3	0.008204	30	0.706316	80	0.999914
4	0.016356	35	0.814383	100	1.
5	0.027136	40	0.891232	150	1.
10	0.116948	45	0.940976	200	1.
15	0.252901	50	0.970374	250	1.
20	0.411438	60	0.994123	300	1.

Figure 1.3 gives a graph of this function. It is a rather surprising fact that $p(23) = 0.507297$, so that if you randomly choose a set of 23 people on earth

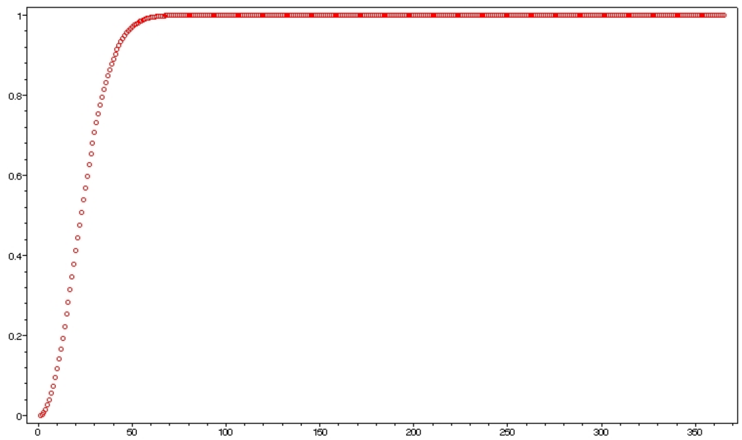


Figure 1.3: The probability of a common birthday among n people.

then there is a slightly better than 50% chance that at least two of them will have the same birthday. (Approximately – we have ignored leap years and twins.)

1.2.3 An example with multisets.

Example 1.21. A packet of *Maynard's Wine Gums* consists of a roll or packet of 10 candies, each of which has one of five "flavours" – *Green, Yellow, Orange, Red*, or *Purple*. I especially like the purple ones. What is the chance that when I buy some Wine Gums there are exactly k purple candies (for each $0 \leq k \leq 10$)?

This example is designed to illustrate the fact that the probabilities depend on which model is used to analyze the situation. There are two reasonable possibilities for this problem, which I will call the **dice model** and the **multiset model**.

In the "dice model" we keep track of the fact that the candies are stacked up in the roll from bottom to top, so there is a natural sequence $(c_1, c_2, ..., c_{10})$ of flavours one sees when the roll is opened. For example, the sequences

$$(G, P, R, Y, Y, G, O, R, Y, O)$$

and

$$(Y, G, O, P, R, R, Y, G, O, Y)$$

count as different outcomes in this model. We have a sequence of 10 candies, and a choice of one of 5 flavours for each candy, giving a total of $5^{10} = 9765625$ outcomes. (This is equivalent to rolling a sequence of ten 5-sided dice, hence the name for the model.)

In the "multiset model" we disregard the order in which the candies occur in the packet as being an inessential detail. The only important information about the packet is the number of candies of each type that it contains. For example, both of the outcomes in the previous paragraph reduce to the same multiset

$$\{G, G, Y, Y, Y, O, O, R, R, P\}$$

or $(2, 3, 2, 2, 1)$ in this model. Thus we are regarding the packet as a multiset of size 10 with 5 types of element, giving a total of $\binom{10+5-1}{5-1} = \binom{14}{4} = 1001$ outcomes.

Notice that the number of outcomes in the dice model is vastly larger than in the multiset model. It should come as no surprise, then, that the probabilities we compute depend strongly on which of these two models we consider. (The true values for the probabilities depend on the details of the manufacturing process by which the rolls or packets are made. These cannot be calculated, but must be measured instead.)

Consider the dice model first, and let $d(k)$ denote the probability of getting exactly k purple candies in a roll. There are $\binom{10}{k}$ choices for the positions of these k purple candies, and $(5-1)^{10-k}$ choices for the sequence of (non–purple) flavours of the other $10-k$ candies. This gives a total of $\binom{10}{k}4^{10-k}$ outcomes with exactly k purple candies in this model. Therefore,

$$d(k) = \binom{10}{k}\frac{4^{10-k}}{5^{10}}$$

for each $0 \leq k \leq 10$.

Next let's consider the multiset model, and let $m(k)$ denote the probability of getting exactly k purple candies in a packet. If we have k purple candies then the rest of the candies form a multiset of size $10-k$ with elements of 4 types, so there are $\binom{10-k+4-1}{4-1} = \binom{13-k}{3}$ such outcomes in this model. Therefore,

$$m(k) = \frac{\binom{13-k}{3}}{\binom{14}{4}}$$

for each $0 \leq k \leq 10$.

Here is a table of these probabilities (rounded to six decimal places).

k	$d(k)$	$m(k)$
0	0.107374	0.285714
1	0.268435	0.219780
2	0.301990	0.164835
3	0.201327	0.119880
4	0.088080	0.083916
5	0.026424	0.055944
6	0.005505	0.034965
7	0.000786	0.019980
8	0.000074	0.009990
9	0.000004	0.003996
10	0.000000	0.000999

The differences between the two models are clearly seen.

In closing, here are two more points about these models.

First, given a multiset $(m_1, ..., m_t)$ of size n with elements of t types, the number of outcomes in the dice model which "reduce" to this multiset is

$$\binom{n}{m_1, ..., m_t} = \frac{n!}{m_1! \cdot m_2! \cdots m_t!},$$

called a *multinomial coefficient*. This can be seen intuitively by arranging the n elements of the multiset in a line in one of $n!$ ways, and noticing that since we can't tell the m_i elements of type i apart we can freely rearrange them in $m_i!$ ways without changing the arrangement. One could prove this more carefully by induction on t, using the case $t = 2$ of binomial coefficients as part of the induction step. Or, one could give a combinatorial proof by constructing a bijection which makes the informal argument above more precise.

For the second point, the above analysis of the multiset model can be generalized to prove Exercise 1.11: for any integers $n \geq 0$ and $t \geq 2$,

$$\binom{n+t-1}{t-1} = \sum_{k=0}^{n} \binom{n-k+t-2}{t-2}.$$

1.2.4 Poker hands.

Poker is played with a *standard deck* of 52 cards, divided into four *suits*:
spades $\spadesuit$, hearts $\heartsuit$, diamonds $\diamondsuit$, and clubs $\clubsuit$.
Within each suit are 13 cards of different *values*:
A (Ace), 2, 3, 4, 5, 6, 7, 8, 9, 10, J (Jack), Q (Queen), K (King).
An Ace can be *high* (above K) or *low* (below 2) at the player's choice.

Many variations on the game exist, but the common theme is to make the best five-card hand according to the ranking of poker hands. This order is determined by how unlikely it is to be dealt such a hand. From best to worst, the types of poker hand are as follows:

- Straight Flush: five cards of the same suit with consecutive values.
  For example, $8\heartsuit\ 9\heartsuit\ 10\heartsuit\ J\heartsuit\ Q\heartsuit$.
- Four of a Kind (or Quads): four cards of the same value, with any fifth card. For example, $7\spadesuit\ 7\heartsuit\ 7\diamondsuit\ 7\clubsuit\ 4\diamondsuit$.
- Full House (or Tight, or Boat): three cards of the same value, and a pair of cards of another value. For example, $9\spadesuit\ 9\heartsuit\ 9\diamondsuit\ A\diamondsuit\ A\clubsuit$.
- Flush: five cards of the same suit, but not with consecutive values.
  For example, $3\heartsuit\ 7\heartsuit\ 10\heartsuit\ J\heartsuit\ K\heartsuit$.
- Straight: five cards with consecutive values, but not of the same suit.
  For example, $8\heartsuit\ 9\clubsuit\ 10\spadesuit\ J\heartsuit\ Q\diamondsuit$.
- Three of a Kind (or Trips): three cards of the same value, and two more cards not of the same value. For example, $8\spadesuit\ 8\heartsuit\ 8\diamondsuit\ K\diamondsuit\ 5\clubsuit$.
- Two Pair: this is self-explanatory.
  For example, $J\heartsuit\ J\clubsuit\ 6\diamondsuit\ 6\clubsuit\ A\spadesuit$.
- One Pair: this is also self-explanatory.
  For example, $Q\spadesuit\ Q\diamondsuit\ 8\diamondsuit\ 7\clubsuit\ 2\spadesuit$.
- Busted Hand: anything not covered above.
  For example, $K\spadesuit\ Q\diamondsuit\ 8\diamondsuit\ 7\clubsuit\ 2\spadesuit$.

There are $\binom{52}{5} = 2598960$ possible 5-element subsets of a standard deck of 52 cards, so this is the total number of possible poker hands. How many of these hands are of each of the above types? The answers can be found easily on the WWWeb, so there's no sense trying to keep them secret. Here they are: N is the number of outcomes of each type, and $p = N/\binom{52}{5}$ is the probability of each type of outcome (rounded to six decimal places).

Hand	N	p
Straight Flush	40	0.000015
Quads	624	0.000240
Full House	3744	0.001441
Flush	5108	0.001965
Straight	10200	0.003925
Trips	54912	0.021128
Two Pair	123552	0.047539
One Pair	1098240	0.422569
Busted	1302540	0.501177

The derivation of these numbers is excellent practice (see Exercise 1.14), so we will do only two of the cases – Straight, and Busted – as illustrations.

Example 1.22.

- To construct a Straight hand there are 10 choices for the consecutive values of the cards ($A2345$, 23456, ... up to $10JQKA$), and 4^5 choices for the suits on the cards. However, four of these choices for suits give all five cards the same suit – these lead to straight flushes and must be discounted. Hence the total number of straights is $10 \cdot (4^5 - 4) = 10200$.
- To construct a Busted hand there are $\binom{13}{5} - 10$ choices for 5 values of cards which are not consecutive (no straight) and have no pairs. Having chosen these values there are $4^5 - 4$ choices for the suits on the cards which do not give all five cards the same suit (no flush). Hence the total number of busted hands is $\left(\binom{13}{5} - 10\right) \cdot (4^5 - 4) = 1302540$.

1.2.5 Derangements.

Here is a "classical" example. (In this context, the word "derangement" makes more sense in French than in English.)

Example 1.23 (The Derangement Problem)**.** A group of eight people meet for dinner at a fancy restaurant and check their coats at the door. After a delicious gourmet meal the group leaves, and on the way out the eight coats are returned to the eight people completely at random by an incompetent clerk. What is the probability that no-one gets the correct coat?

Of course, we want to solve the derangement problem for any number of people, not just for eight. To state the problem mathematically, list the people P_1, P_2, ..., P_n in any order. We can record who gets whose coat by a sequence of numbers $(c_1, c_2, ..., c_n)$ in which $c_i = j$ means that P_i was given the coat belonging to P_j. The sequence $(c_1, c_2, ..., c_n)$ will thus contain each of the numbers $1, 2, ..., n$ exactly once in some order. In other words, $(c_1, ..., c_n)$ is a permutation of the set $\{1, 2, ..., n\}$, and we assume that this permutation is chosen randomly by the incompetent clerk. Person i gets the correct coat exactly when $c_i = i$. Thus, in general the derangement problem is to determine, for a random permutation $(c_1, ..., c_n)$ of $\{1, 2, ..., n\}$, the probability that $c_i \neq i$ for all $1 \leq i \leq n$.

For small values of n the derangement problem can be analyzed directly, but complications arise as n gets larger. In fact, this example is perfectly designed to illustrate the principle of Inclusion/Exclusion. To see how this applies, for each $1 \leq i \leq n$ let A_i be the set of permutations of $\{1, ..., n\}$ such that $c_i = i$. That is, A_i is the set of ways in which the coats are returned and person P_i gets the correct coat. From that interpretation, the union of sets $A_1 \cup A_2 \cup \cdots \cup A_n$ is the set of ways in which the coats are returned and **at least one person** gets the correct coat. Therefore, the complementary set of permutations gives those ways of returning the coats so that **no-one** gets the correct coat. The number of these *derangements* of n objects is thus

$$n! - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

It remains to apply Inclusion/Exclusion to determine $|A_1 \cup \cdots \cup A_n|$. To do this we need to determine $|A_S|$ for every nonempty subset $\varnothing \neq S \subseteq$

$\{1, 2, ..., n\}$. Consider the example $n = 8$ and $S = \{2, 3, 6\}$. In this case $A_{\{2,3,6\}} = A_2 \cap A_3 \cap A_6$ is the set of those permutations of $\{1, ..., 8\}$ such that $c_2 = 2$ and $c_3 = 3$ and $c_6 = 6$. Such a permutation looks like $\square\ 2\ 3\ \square\ \square\ 6\ \square\ \square$ in which the boxes are filled with the numbers $\{1, 4, 5, 7, 8\}$ in some order. Since there are $5!$ lists of the set$\{1, 4, 5, 7, 8\}$ it follows that $|A_{\{2,3,6\}}| = 5! = 120$ in this case. The general case is similar. If $\varnothing \neq S \subseteq \{1, 2, ..., n\}$ is a k-element subset then the permutations of $\{1, 2, ..., n\}$ in A_S are obtained by fixing $c_i = i$ for all $i \in S$, and then listing the remaining $n - k$ elements of $\{1, ..., n\} \setminus S$ in the remaining spaces. Since there are $(n - k)!$ such lists we see that $|A_S| = (n - k)!$.

Since $|A_S| = (n - k)!$ for every k-element subset of $\{1, 2, ..., n\}$, and there are $\binom{n}{k}$ such k-element subsets, Inclusion/Exclusion implies that

$$|A_1 \cup \cdots \cup A_n| \ = \ \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} (n-k)! \ = \ n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}.$$

It follows that the number of derangements of n objects is

$$n! - n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \ = \ n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Since the total number of permutations of n objects is $n!$, the probability that a randomly chosen permutation of $\{1, 2, ..., n\}$ is a derangement is

$$D_n \ = \ \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

The following table lists the first several values of the function D_n (with the decimals rounded to six places). Notice that for $n \geq 7$ the value of D_n changes very little. If you recall the Taylor series expansion of the exponential function

$$\mathrm{e}^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

then it is easy to see that as n tends to infinity the probability D_n approaches the limiting value of $e^{-1} = 0.3678794412...$.

n	D_n	D_n
0	$1/1$	1.
1	$0/1$	0.000000
2	$1/2$	0.500000
3	$1/3$	0.333333
4	$3/8$	0.375000
5	$11/30$	0.366667
6	$53/144$	0.368056
7	$103/280$	0.367857
8	$2119/5760$	0.367882
9	$16687/45630$	0.367879
10	$16481/44800$	0.367879

In summary, for the original Example 1.23, the probability that no-one gets their own coat is very close to 36.79%.

1.3 Exercises.

Exercise 1.1. Fix integers $n \geq 0$ and $t \geq 1$. Consider a randomly chosen multiset of size n with elements of t types. For each part below, calculate the probability that the multiset has the stated property, and give a brief explanation.

(a) Every type of element occurs at most once.
(b) Every type of element occurs at least once.
(c) Every type of element occurs an even number of times.
(d) Every type of element occurs an odd number of times.
(e) For $k \in \mathbb{N}$, exactly k types of element occur with multiplicity at least one.
(f) For $k \in \mathbb{N}$, exactly k types of element occur with multiplicity at least two.

Exercise 1.2. Consider rolling six fair 6-sided dice, which are distinguishable, so that there are $6^6 = 46656$ equally likely outcomes. Count how many outcomes are of each of the following types. (The answers add up to 46656.)

(a) Six-of-a-kind.
(b) Five-of-a-kind and a single.
(c) Four-of-a-kind and a pair.
(d) Four-of-a-kind and two singles.
(e) Two triples.
(f) A triple, a pair, and a single.
(g) A triple and three singles.
(h) Three pairs.
(i) Two pairs and two singles.
(j) One pair and four singles.
(k) Six singles.

Exercise 1.3. Let $m \geq 1$, $d \geq 2$, and $k \geq 0$ be integers. When rolling m fair dice, each of which has d sides, what is the probability of rolling exactly k pairs and $m - 2k$ singles?

Exercise 1.4.

(a) Prove that $\rightleftharpoons$ is an equivalence relation.
(b) Prove Proposition 1.11.

Exercise 1.5. Define a function $f : \mathbb{Z} \to \mathbb{N}$ as follows: for $a \in \mathbb{Z}$,

$$f(a) = \begin{cases} 2a & \text{if } a \geq 0, \\ -1 - 2a & \text{if } a < 0. \end{cases}$$

Show that $f : \mathbb{Z} \to \mathbb{N}$ is a bijection as follows.

(a) Define a function $g : \mathbb{N} \to \mathbb{Z}$.
(b) Show that for all $a \in \mathbb{Z}$, $g(f(a)) = a$.
(c) Show that for all $b \in \mathbb{N}$, $f(g(b)) = b$.

Exercise 1.6. Complete the proof in Example 1.13.

Exercise 1.7. Give bijective proofs of the following identities.

(a) For all $n \in \mathbb{N}$, $\sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1}$.
(b) For all $n \in \mathbb{N}$, $\sum_{k=0}^{n} \binom{n}{k} k(k-1) = n(n-1)2^{n-2}$.

Exercise 1.8. For an integer $n \geq 1$, give a bijective proof that

$$\sum_{\text{even } k} \binom{n}{k} = \sum_{\text{odd } k} \binom{n}{k}$$

Exercise 1.9. Let n be a positive integer. Let S_n be the set of all ordered pairs of sets (A, B) in which $A \subseteq B \subseteq \{1, 2, ..., n\}$. Let T_n be the set of all functions $f : \{1, 2, ..., n\} \to \{1, 2, 3\}$.

(a) What is $|T_n|$? (Explain.)
(b) Define a bijection $g : S_n \to T_n$. Explain why $g((A, B)) \in T_n$ for any $(A, B) \in S_n$. (You do not need to explain why g is a bijection.)
(c) Define the inverse function $g^{-1} : T_n \to S_n$ of your bijection g from part (b). (You do not need to explain why g and g^{-1} are mutually inverse bijections.)
(d) By counting S_n and T_n in two different ways, deduce that

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

Exercise 1.10. Fix integers $n \geq 0$ and $k \geq 1$. Let $\mathcal{A}(n, k)$ be the set of sequences $(a_1, ..., a_k) \in \mathbb{N}^k$ such that $a_1 + \cdots + a_k = n$, and j divides a_j for all $1 \leq j \leq k$. Let $\mathcal{B}(n, k)$ be the set of sequences $(b_1, ..., b_k) \in \mathbb{N}^k$ such that $b_1 + \cdots + b_k = n$, and $b_1 \geq b_2 \geq \cdots \geq b_k$. For example, here are the sets for $n = 7$ and $k = 3$:

$\mathcal{A}(7,3)$	$\mathcal{B}(7,3)$
$(7,0,0)$	$(7,0,0)$
$(5,2,0)$	$(6,1,0)$
$(4,0,3)$	$(5,2,0)$
$(3,4,0)$	$(4,3,0)$
$(2,2,3)$	$(5,1,1)$
$(1,6,0)$	$(4,2,1)$
$(1,0,6)$	$(3,3,1)$
$(0,4,3)$	$(3,2,2)$

Construct a pair of mutually inverse bijections between the sets $\mathcal{A}(n,k)$ and $\mathcal{B}(n,k)$.

Exercise 1.11. For integers $n \geq 0$ and $t \geq 2$, give a bijective proof that

$$\binom{n+t-1}{t-1} = \sum_{k=0}^{n} \binom{n-k+t-2}{t-2}.$$

Exercise 1.12. For integers $n \geq 1$ and $t \geq 1$, give a bijective proof that

$$\binom{n+t-1}{t-1} = \sum_{k=0}^{t} \binom{t}{k}\binom{n-1}{k-1}.$$

Exercise 1.13. Choose a permutation σ of $\{1, 2, ..., 7\}$ at random, so that each of the $7! = 5040$ permutations are equally likely. What are the probabilities of the following events?

1. Numbers 1 and 2 are consecutive (*i.e.* 3672154 or 3671254).
2. Number 1 is to the left of 2.
3. No two odd numbers are consecutive.
4. Any other condition you can think of.
5. Any similar questions for permutations of $\{1, 2, ..., n\}$.

Exercise 1.14. (The $r = 13$ and $s = 4$ case of this exercise completes the table in Subsection 1.2.4.) Let $r \geq 2$ and $s \geq 2$ be integers. Consider a (non-standard) deck of rs cards, divided into s suits each with cards of r different values. The cards in each suit are numbered A, 2, 3, ..., r, and A can be either below 2 or above r. Choose five cards from such a deck in one of $\binom{rs}{5}$ ways. How many ways are there to produce each kind of hand for this "poker in an alternate universe"?

(a) Count "quints" (five-of-a-kinds).
(b) Count straight flushes.
(c) Count quads.
(d) Count full houses.
(e) Count flushes.
(f) Count straights.
(g) Count trips.
(h) Count two-pairs.
(i) Count one-pairs.
(j) Count busted hands.

Exercise 1.15. The game called “Crowns and Anchors” or “Birdcage” was popular on circus midways early in the 20th century. It is a game between a Player and the House, played as follows. First, the Player wagers w dollars on an integer p from one to six. Next, the House rolls three six-sided dice. For every die that shows p dots on top, the House pays the Player w dollars, but if no dice show p dots on top then the Player’s wager is forfeited, and goes to the House. (Assume that the dice are fair, so that every outcome is equally likely.)

For example, if I wager two dollars on the number five, and the dice show five, five, and three dots, respectively, then the House pays me four dollars for a total of six (a profit of four dollars). However, if in this case the dice show four, three, and two dots, respectively, then the House takes my wager for a total of zero (a loss of two dollars).

(a) For every dollar that the Player wagers, how much money should the Player expect to win back in the long run? Would you play this game?
(b) In a parallel universe there is a game of Crowns and Anchors being played with $m \geq 1$ dice, each of which has $d \geq 2$ sides. (Assume that the dice are fair, so that every outcome is equally likely.) In which universes does the Player win in the long run? In which universes does the House win in the long run? In which universes is the game completely fair?

Chapter 2

The Idea of Generating Series.

We will be dealing algebraically with infinite power series $G(x) = \sum_{n=0}^{\infty} g_n x^n$ in which the coefficients $\mathbf{g} = (g_0, g_1, g_2, ...)$ form a sequence of integers. These can, for the most part, be handled just like polynomials. Problems of convergence can arise if one tries to substitute a particular real or complex value for the indeterminate x. We usually don't do this, and finiteness of all the coefficients of $G(x)$ is all that we require.

Example 2.1 (The Geometric Series)**.** The simplest infinite case of power series is if all the coefficients equal one. Then

$$G = 1 + x + x^2 + x^3 + x^4 + \cdots .$$

Multiply this by x:

$$xG = x + x^2 + x^3 + x^4 + \cdots .$$

It follows that $G - xG = (1 - x)G = 1$. In conclusion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

Don't worry about convergence – this is just algebra!

2.1 The Binomial Theorem and Binomial Series.

We develop two of the most useful facts that we will need in what follows. The proofs are also good illustrations of calculating with generating series.

Theorem 2.2 (The Binomial Theorem)**.** *For any natural number $n \in \mathbb{N}$,*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

This formula is an identity between two polynomials in the variable x. You probably have seen a proof of it by induction on n, but we are going to prove it here using the bijection between subsets and indicator vectors discussed in Example 1.12.

Proof. Recall that $\mathcal{P}(n)$ is the set of all subsets of $\{1, 2, ..., n\}$, and that $\{0, 1\}^n$ is the set of all indicator vectors $\alpha = (a_1, a_2, ..., a_n)$ in which each coordinate is either 0 or 1. Example 1.12 gives a bijection between these two sets, which you should recall. For example, when $n = 8$ the subset $\{2, 3, 5, 7\}$ corresponds to the indicator vector $(0, 1, 1, 0, 1, 0, 1, 0)$. The constructions $S \mapsto \alpha(S)$ and $\alpha \mapsto S(\alpha)$ are mutually inverse bijections between the sets $\mathcal{P}(n)$ and $\{0, 1\}^n$. From this, we concluded that $|\mathcal{P}(n)| = |\{0, 1\}^n| = 2^n$, but we can deduce more. Notice that if S is a subset with k elements then it corresponds to an indicator vector α that sums to k. It is sometimes helpful to record this information in a little table, like this:

$$\begin{aligned} \mathcal{P}(n) &\rightleftharpoons \{0,1\}^n \\ S &\leftrightarrow \alpha = (a_1, a_2, ..., a_n) \\ |S| &= a_1 + a_2 + \cdots + a_n. \end{aligned}$$

Because of this bijection, if we introduce an "indeterminate" x, and if S corresponds to α, then

$$x^{|S|} = x^{a_1+a_2+\cdots+a_n}.$$

Moreover, also because of this bijection, summing over all subsets is equivalent to summing over all indicator functions. That is,

$$\sum_{S \in \mathcal{P}(n)} x^{|S|} = \sum_{\alpha \in \{0,1\}^n} x^{a_1+a_2+\cdots+a_n}.$$

Now we can simplify both sides separately. On the LHS, we know from Theorem 1.5 that there are $\binom{n}{k}$ k-element subsets of an n-element set, for each $0 \leq k \leq n$. Therefore,

$$\sum_{S \in \mathcal{P}(n)} x^{|S|} = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

On the RHS, summing over all the indicator vectors $\alpha \in \{0,1\}^n$ is equivalent to summing over all $a_1 \in \{0,1\}$ and all $a_2 \in \{0,1\}$ and so on,... until all $a_n \in \{0,1\}$. This gives

$$\sum_{\alpha \in \{0,1\}^n} x^{a_1+a_2+\cdots+a_n} = \sum_{a_1=0}^{1} \sum_{a_2=0}^{1} \cdots \sum_{a_n=0}^{1} x^{a_1+a_2+\cdots+a_n}$$
$$= \sum_{a_1=0}^{1} x^{a_1} \sum_{a_2=0}^{1} x^{a_2} \cdots \sum_{a_n=0}^{1} x^{a_n} = \left(\sum_{a=0}^{1} x^a \right)^n = (1+x)^n.$$

This proves the Binomial Theorem. With practice and familiarity, it becomes a one-line proof:

$$\sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{S \in \mathcal{P}(n)} x^{|S|} = \sum_{\alpha \in \{0,1\}^n} x^{a_1+a_2+\cdots+a_n} = (1+x)^n.$$

□

Example 2.3 (Vandermonde Convolution.)**.** As mentioned in Subsection 1.2.1, the Binomial Theorem easily implies the Vandermonde Convolution formula. To see this, begin with the obvious identity of polynomials

$$(1+x)^{m+n} = (1+x)^m \cdot (1+x)^n$$

and use the Binomial Theorem to expand each of the factors.

$$\sum_{k=0}^{m+n} \binom{m+n}{k} x^k = \left(\sum_{j=0}^{m} \binom{m}{j} x^j \right) \left(\sum_{i=0}^{n} \binom{n}{i} x^i \right)$$
$$= \sum_{j=0}^{m} \sum_{i=0}^{n} \binom{m}{j} \binom{n}{i} x^{j+i} = \sum_{k=0}^{m+n} \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j} x^k.$$

(The last step is accomplished by re-indexing the double summation.) Since the polynomials on the LHS and on the RHS are equal, they must have the same coefficients. By comparing the coefficients of x^k on both sides we see that

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j},$$

giving the result.

Consider the set $\mathcal{M}(t)$ of all multisets with $t \geq 1$ types of elements, regardless of the size of the multiset. That is, an element of $\mathcal{M}(t)$ is a sequence $\mu = (m_1, m_2, ..., m_t)$ of t natural numbers, and the size of the multiset is $|\mu| = m_1 + m_2 + \cdots + m_t$. By Theorem 1.9, for each $n \in \mathbb{N}$ there are $\binom{n+t-1}{t-1}$ elements of $\mathcal{M}(t)$ of size n. By analogy with the Binomial Theorem 2.2, we could collect these numbers as the coefficients of a power series:

$$\sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n.$$

The Binomial Series is an algebraic formula for this summation.

Theorem 2.4 (The Binomial Series)**.** *For any positive integer $t \geq 1$,*

$$\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n.$$

(Properly, this is the binomial series with negative integer exponent. The general binomial series is discussed in Subsection 4.4.1).

Proof. The key observation is that the set of all multisets with $t \geq 1$ types of elements is $\mathcal{M}(t) = \mathbb{N}^t$, the Cartesian product of t copies of the natural numbers $\mathbb{N}$. This leads to a calculation similar to the proof of the Binomial Theorem above, based on this structure:

$$\begin{aligned} \mathcal{M}(t) &= \mathbb{N}^t \\ \mu &= (m_1, .., m_t) \\ |\mu| &= m_1 + \cdots + m_t \end{aligned}$$

We use this to calculate as follows. The first equality is by Theorem 1.9.

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{n+t-1}{t-1}x^n &= \sum_{n=0}^{\infty}|\mathcal{M}(n,t)|\,x^n = \sum_{\mu\in\mathcal{M}(t)} x^{|\mu|}\\ &= \sum_{(m_1,m_2,\ldots,m_t)\in\mathbb{N}^t} x^{m_1+m_2+\cdots m_t} = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_t=0}^{\infty} x^{m_1+m_2+\cdots+m_t}\\ &= \sum_{m_1=0}^{\infty}x^{m_1}\sum_{m_2=0}^{\infty}x^{m_2}\cdots\sum_{m_t=0}^{\infty}x^{m_t} = \left(\sum_{m=0}^{\infty}x^m\right)^t = \frac{1}{(1-x)^t}.\end{aligned}$$

The last equality is by the geometric series $1+x+x^2+\cdots = 1/(1-x)$. $\square$

2.2 The Theory in General.

In general terms we have a sequence of numbers $\mathrm{g} = (g_0, g_1, g_2, \ldots)$ which we would like to determine. To do this we introduce an *indeterminate* x and encode these numbers as the coefficients of a power series

$$G(x) \;=\; g_0 + g_1x + g_2x^2 + g_3x^3 + \cdots \;=\; \sum_{n=0}^{\infty} g_nx^n,$$

called the *generating series* for the sequence g.

By "indeterminate" we mean that x behaves algebraically as if it were a number, but that it does not have any particular value. It should be thought of as a punctuation mark that is there to keep the different coefficients g_n separated from each other. Sometimes, people call x a "variable" – but that word is meant to convey the idea that x is a number, but that we don't know specifically what the value of that number is. The word "indeterminate" is meant to convey the idea that x *does not have any value at all* – it is just a punctuation mark. As will be seen, it is the coefficients of these power series that carry information about our counting problems.

In this chapter and the next we will see how to use this strategy to encode the answers to various counting problems as generating series. In Chapter 4 we will see how to get numbers out of these power series in order to answer the counting problems explicitly.

2.2.1 Generating series.

Let $\mathcal{A}$ be a set of "objects" which we want to count. For example, $\mathcal{A}$ might be the set of subsets of the set $\{1, 2, ..., n\}$. Or, $\mathcal{A}$ might be the set of all multisets with t types of elements. The set $\mathcal{A}$ can be quite arbitrary, but we assume that each element of $\mathcal{A}$ has a "size" or "weight" attached to it. The weight of $\alpha \in \mathcal{A}$ is a nonnegative integer $\omega(\alpha) \in \mathbb{N}$. We just require that there are only finitely many elements of $\mathcal{A}$ of any given weight.

Definition 2.5 (Weight Function)**.** Let $\mathcal{A}$ be a set. A function $\omega : \mathcal{A} \to \mathbb{N}$ from $\mathcal{A}$ to the set $\mathbb{N}$ of natural numbers is a *weight function* provided that for all $n \in \mathbb{N}$, the set

$$\mathcal{A}_n = \omega^{-1}(n) = \{\alpha \in \mathcal{A} : \ \omega(\alpha) = n\}$$

is finite. That is, for every $n \in \mathbb{N}$ there are only finitely many elements $\alpha \in \mathcal{A}$ of weight n.

Notice that if $\mathcal{A}$ is a set with a weight function $\omega : \mathcal{A} \to \mathbb{N}$, then

$$\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n$$

is a (disjoint) union of countably many finite sets, and so $\mathcal{A}$ is itself either finite or countably infinite.

Definition 2.6 (Generating series)**.** Let $\mathcal{A}$ be a set with a weight function $\omega : \mathcal{A} \to \mathbb{N}$ as in Definition 2.5. The *generating series of $\mathcal{A}$ with respect to ω* is

$$A(x) = \Phi^{\omega}_{\mathcal{A}}(x) = \sum_{\alpha \in \mathcal{A}} x^{\omega(\alpha)}.$$

(We usually suppress the superscript from the notation.) Remember – the indeterminate x does not have a value. It is just used to keep track of the weight of each object $\alpha \in \mathcal{A}$ in the exponent.

Proposition 2.7. *Let $\mathcal{A}$ be a set with a weight function $\omega : \mathcal{A} \to \mathbb{N}$, and let*

$$\Phi_{\mathcal{A}}(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

For every $n \in \mathbb{N}$, the number of elements of $\mathcal{A}$ of weight n is $a_n = |\mathcal{A}_n|$.

Proof.

$$\Phi_{\mathcal{A}}(x) = \sum_{\alpha \in \mathcal{A}} x^{\omega(\alpha)} = \sum_{n=0}^{\infty} \sum_{\alpha \in \mathcal{A}:\ \omega(\alpha)=n} x^{\omega(\alpha)} = \sum_{n=0}^{\infty} x^n \sum_{\alpha \in \mathcal{A}_n} 1 = \sum_{n=0}^{\infty} |\mathcal{A}_n|\, x^n.$$

Thus, for each $n \in \mathbb{N}$, the coefficient of x^n in $\Phi_{\mathcal{A}}(x)$ is the number of elements in $\mathcal{A}$ that have weight n. □

Since we will be doing a lot of long calculations with power series, and because of Proposition 2.7, it is useful to have a handy notation for extracting coefficients from them.

Definition 2.8. Let $G(x) = g_0 + g_1 x + g_2 x^2 + \cdots = \sum_{n=0}^{\infty} g_n x^n$ be any power series. Then for any $k \in \mathbb{N}$,

$$[x^k]\, G(x) \;=\; g_k$$

is the coefficient of x^k in the power series $G(x)$.

Example 2.9. For example, for any natural numbers $a, b \in \mathbb{N}$,

$$[x^a]\frac{1}{(1-x)^{1+b}} = [x^a] \sum_{n=0}^{\infty} \binom{n+b}{b} x^n = \binom{a+b}{b}.$$

2.2.2 The Sum, Product, and String Lemmas.

Lemma 2.10 (The Sum Lemma.). *Let $\mathcal{A}$ and $\mathcal{B}$ be disjoint sets, so that $\mathcal{A} \cap \mathcal{B} = \varnothing$. Assume that $\omega : (\mathcal{A} \cup \mathcal{B}) \to \mathbb{N}$ is a weight function on the union of $\mathcal{A}$ and $\mathcal{B}$. We may regard ω as a weight function on each of $\mathcal{A}$ or $\mathcal{B}$ separately (by restriction). Under these conditions,*

$$\Phi_{\mathcal{A}\cup\mathcal{B}}(x) = \Phi_{\mathcal{A}}(x) + \Phi_{\mathcal{B}}(x).$$

Proof. From the definition of generating series,

$$\Phi_{\mathcal{A}\cup\mathcal{B}}(x) \;=\; \sum_{\sigma\in\mathcal{A}\cup\mathcal{B}} x^{\omega(\sigma)} \;=\; \sum_{\sigma\in\mathcal{A}} x^{\omega(\sigma)} + \sum_{\sigma\in\mathcal{B}} x^{\omega(\sigma)} \;=\; \Phi_{\mathcal{A}}(x) + \Phi_{\mathcal{B}}(x).$$

(The condition that $\mathcal{A} \cap \mathcal{B} = \varnothing$ is needed for the second equality.) □

In fact, the above proof can be generalized slightly to give more.

Lemma 2.11 (The Infinite Sum Lemma.). *Let $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$,... be pairwise disjoint sets (so that $\mathcal{A}_i \cap \mathcal{A}_j = \varnothing$ if $i \neq j$), and let $\mathcal{B} = \bigcup_{j=0}^{\infty} \mathcal{A}_j$. Assume that $\omega : \mathcal{B} \to \mathbb{N}$ is a weight function. We may regard ω as a weight function on each of the sets $\mathcal{A}_j$ separately (by restriction). Under these conditions,*

$$\Phi_{\mathcal{B}}(x) = \sum_{j=0}^{\infty} \Phi_{\mathcal{A}_j}(x).$$

Proof. Exercise 2.6. □

Lemma 2.12 (The Product Lemma.). *Let $\mathcal{A}$ and $\mathcal{B}$ be sets with weight functions $\omega : \mathcal{A} \to \mathbb{N}$ and $\nu : \mathcal{B} \to \mathbb{N}$, respectively. Define $\eta : \mathcal{A} \times \mathcal{B} \to \mathbb{N}$ by putting $\eta(\alpha, \beta) = \omega(\alpha) + \nu(\beta)$ for all $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$. Then η is a weight function on $\mathcal{A} \times \mathcal{B}$, and*

$$\Phi^{\eta}_{\mathcal{A}\times\mathcal{B}}(x) = \Phi^{\omega}_{\mathcal{A}}(x) \cdot \Phi^{\nu}_{\mathcal{B}}(x).$$

(The superscripts ω, ν, and η indicate which weight function is being used for each set.)

Proof. To see that η is a weight function, consider any $n \in \mathbb{N}$. There are $n+1$ choices for an integer $0 \leq k \leq n$. For each such k, there are only finitely many pairs $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$ with $\omega(\alpha) = k$ and $\omega(\beta) = n - k$. That is, the set of elements of $\mathcal{A} \times \mathcal{B}$ of weight n is

$$(\mathcal{A} \times \mathcal{B})_n = \bigcup_{k=0}^{n} \mathcal{A}_k \times \mathcal{B}_{n-k},$$

a finite (disjoint) union of finite sets. It follows that there are only finitely many elements of $\mathcal{A} \times \mathcal{B}$ of weight n. Now,

$$\begin{aligned}\Phi^{\eta}_{\mathcal{A}\times\mathcal{B}}(x) &= \sum_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} x^{\eta(\alpha,\beta)} = \sum_{\alpha\in\mathcal{A}} \sum_{\beta\in\mathcal{B}} x^{\omega(\alpha)+\nu(\beta)} \\ &= \sum_{\alpha\in\mathcal{A}} x^{\omega(\alpha)} \cdot \sum_{\beta\in\mathcal{B}} x^{\nu(\beta)} = \Phi_{\mathcal{A}}(x) \cdot \Phi_{\mathcal{B}}(x).\end{aligned}$$

□

The Product Lemma 2.12 can be extended to the Cartesian product of any finite number of sets, by induction on the number of factors. (Exercise 2.7.)

Finally, the String Lemma combines both disjoint union and Cartesian product, as follows. Let $\mathcal{A}$ be a set with a weight function $\omega : \mathcal{A} \to \mathbb{N}$. For any $k \in \mathbb{N}$, the Cartesian product of k copies of $\mathcal{A}$ is denoted by $\mathcal{A}^k$. The entries of $\mathcal{A}^k$ are k-tuples $(\alpha_1, \alpha_2, ..., \alpha_k)$ with each $\alpha_i \in \mathcal{A}$. Notice that $\mathcal{A}^0 = \{\varepsilon\}$ is the one-element set whose only element is the empty string $\varepsilon = ()$ of length zero. We can define a weight function ω_k on $\mathcal{A}^k$ by putting

$$\omega_k(\alpha_1, ..., \alpha_k) = \omega(\alpha_1) + \cdots + \omega(\alpha_k).$$

It is a good exercise to check that this is a weight function. Note that the weight of the empty string is zero. Repeated application of the Product Lemma 2.12 shows that for all $k \in \mathbb{N}$,

$$\Phi_{\mathcal{A}^k}(x) = \left(\Phi_{\mathcal{A}}(x)\right)^k.$$

We can take the union of the sets $\mathcal{A}^k$ for all $k \in \mathbb{N}$:

$$\mathcal{A}^* = \bigcup_{k=0}^{\infty} \mathcal{A}^k.$$

Notice that the sets in this union are pairwise disjoint, since each $\mathcal{A}^k$ consists of strings with exactly k coordinates. We define a function $\omega^* : \mathcal{A}^* \to \mathbb{N}$ by saying that $\omega^* = \omega_k$ when restricted to $\mathcal{A}^k$.

Lemma 2.13. *Let $\mathcal{A}$ be a set with weight function $\omega : \mathcal{A} \to \mathbb{N}$, and define $\mathcal{A}^*$ and $\omega^* : \mathcal{A}^* \to \mathbb{N}$ as above. Then ω^* is a weight function on $\mathcal{A}^*$ if and only if there are no elements in $\mathcal{A}$ of weight zero (that is, $\mathcal{A}_0 = \varnothing$).*

Proof. If $\gamma \in \mathcal{A}$ has weight zero, $\omega(\gamma) = 0$, then for any natural number $k \in \mathbb{N}$, a sequence of k γ-s in $\mathcal{A}^k$ also has weight zero: $\omega_k(\gamma, \gamma, ..., \gamma) = 0$. So, by the way $\omega^* : \mathcal{A}^* \to \mathbb{N}$ is defined, there are infinitely many elements of weight zero in $\mathcal{A}^*$, so that ω^* is not a weight function.

Conversely, assume that every element of $\mathcal{A}$ has weight at least 1. Then, for each $k \in \mathbb{N}$, every element of $\mathcal{A}^k$ has weight at least k. Now consider any $n \in \mathbb{N}$ and all the strings $(\alpha_1, ..., \alpha_k) \in \mathcal{A}^*$ of weight n. By the previous sentence, if there are any such strings of length k then $0 \leq k \leq n$. For each $0 \leq k \leq n$, $\mathcal{A}^k$ has only finitely many elements of weight n. It follows that $\mathcal{A}^*$ has only finitely many elements of weight n. Therefore, ω^* is a weight function on $\mathcal{A}^*$. □

Lemma 2.14 (The String Lemma.)**.** *Let $\mathcal{A}$ be a set with a weight function $\omega : \mathcal{A} \to \mathbb{N}$ such that there are no elements of $\mathcal{A}$ of weight zero. Then*

$$\Phi_{\mathcal{A}^*}(x) = \frac{1}{1 - \Phi_{\mathcal{A}}(x)}.$$

Proof. By the Infinite Sum and Product Lemmas 2.11 and 2.12,

$$\Phi_{\mathcal{A}^*}(x) \;=\; \sum_{k=0}^{\infty} \Phi_{\mathcal{A}^k}(x) \;=\; \sum_{k=0}^{\infty} \left(\Phi_{\mathcal{A}}(x)\right)^k \;=\; \frac{1}{1 - \Phi_{\mathcal{A}}(x)}.$$

□

2.3 Compositions.

Definition 2.15. A *composition* is a finite sequence of positive integers:

$$\gamma = (c_1, c_2, ..., c_k),$$

in which $k \in \mathbb{N}$ is a natural number, and each $c_i \geq 1$ is a positive integer. The entries c_i are called the *parts* of the composition. The *length* of the composition is $\ell(\gamma) = k$, the number of parts. The *size* of the composition is

$$|\gamma| = c_1 + c_2 + \cdots + c_k,$$

the sum of the parts.

Notice that there is exactly one composition of length zero: this is $\varepsilon = ()$, the empty string with no entries. Compositions are related to multisets, but there are two important differences: the parts of a composition must be positive integers, not just nonnegative, and the length of a composition might not be specified, while the number of types of element in a multiset must be fixed.

Example 2.16. Here are all the compositions of size five:

(5)	(2,3)	(2,2,1)	(1,2,1,1)
(4,1)	(3,1,1)	(2,1,2)	(1,1,2,1)
(1,4)	(1,3,1)	(1,2,2)	(1,1,1,2)
(3,2)	(1,1,3)	(2,1,1,1)	(1,1,1,1,1)

In this section we will apply the results of Subsection 2.2.2 to obtain formulas for the generating series of various sets of compositions defined by imposing some extra conditions. In Chapter 4 we will see how to use this information to actually count such things.

Theorem 2.17. *Let $P = \{1, 2, 3, ...\}$ be the set of positive integers.*

(a) The set $\mathcal{C}$ of all compositions is $\mathcal{C} = P^*$.

(b) The generating series for $\mathcal{C}$ with respect to size is

$$\Phi_{\mathcal{C}}(x) = 1 + \frac{x}{1-2x}.$$

(c) For each $n \in \mathbb{N}$, the number of compositions of size n is

$$|\mathcal{C}_n| = \begin{cases} 1 & \text{if } n = 0, \\ 2^{n-1} & \text{if } n \geq 1. \end{cases}$$

Proof. A single part is a positive integer $c \in P = \{1, 2, 3, ...\}$. A composition of length k is a sequence $\gamma = (c_1, c_2, ..., c_k)$ of k positive integers, so is an element of P^k. Since the length k can be any natural number, the set $\mathcal{C}$ of all compositions is

$$\mathcal{C} = \bigcup_{k=0}^{\infty} P^k = P^*.$$

This proves part (a).

The generating series for one-part compositions with respect to size is

$$\Phi_P(x) \;=\; \sum_{c=1}^{\infty} x^c = x + x^2 + x^3 + \cdots \;=\; \frac{x}{1-x},$$

by the geometric series. From the String Lemma 2.14 it follows that

$$\Phi_{\mathcal{C}}(x) \;=\; \Phi_{P^*}(x) \;=\; \frac{1}{1 - x/(1-x)} \;=\; \frac{1-x}{1-2x} \;=\; 1 + \frac{x}{1-2x}.$$

This proves part (b).

Expanding $C(x) = \Phi_{\mathcal{C}}(x)$ using the geometric series, we obtain

$$C(x) \;=\; 1 + \sum_{j=0}^{\infty} 2^j\, x^{j+1} \;=\; 1 + \sum_{n=1}^{\infty} 2^{n-1}\, x^n.$$

Since $|\mathcal{C}_n| = [x^n]C(x)$ is the coefficient of x^n in $C(x)$, this proves part (c). □

Proposition 2.23 is a bijection which gives a combinatorial proof of Theorem 2.17(c).

Many variations on the proof of Theorem 2.17 are possible. We will do a few as examples, and present many more as exercises. The general approach consists of three steps:

- Identify the allowed values for each part. This might depend on the position of the part within the composition.
- Identify the allowed lengths for the compositions.
- Apply the Sum, Product, and String Lemmas to obtain a formula for the generating series.

Example 2.18. Let $\mathcal{F}$ be the set of all compositions in which each part is either one or two.

- The allowed sizes for a part are 1 or 2, so $P = \{1, 2\}$ is the set of allowed parts. The generating series for a single part is $x + x^2$.
- The length can be any natural number $k \in \mathbb{N}$. By the Product Lemma, the generating series for compositions in $\mathcal{F}$ of length k is $(x + x^2)^k$.
- Since $\mathcal{F} = \{1, 2\}^*$, the String Lemma implies that

$$F(x) \;=\; \Phi_{\mathcal{F}}(x) \;=\; \sum_{n=0}^{\infty} f_n\, x^n = \sum_{k=0}^{\infty} (x + x^2)^k = \frac{1}{1 - x - x^2}.$$

Here, $f_n = [x^n]F(x) = |\mathcal{F}_n|$ is the number of compositions in $\mathcal{F}$ of size n. In Section 4.1 we will see how to use this information to get a formula for the numbers f_n.

Example 2.19. Let $\mathcal{H}$ be the set of all compositions in which each part is at least two.

- The allowed sizes for a part are $P = \{2, 3, 4, ...\}$. The generating series for a single part is

$$\Phi_P(x) \;=\; \sum_{c=2}^{\infty} x^c \;=\; x^2 + x^3 + x^4 + \cdots \;=\; \frac{x^2}{1 - x}.$$

- The length can be any natural number $k \in \mathbb{N}$. By the Product Lemma, the generating series for compositions in $\mathcal{H}$ of length k

is

$$\left(\frac{x^2}{1-x}\right)^k.$$

- Since $\mathcal{H} = P^*$, the String Lemma implies that

$$\begin{aligned} H(x) \;&=\; \Phi_{\mathcal{H}}(x) \;=\; \sum_{n=0}^{\infty} h_n\, x^n \;=\; \sum_{k=0}^{\infty} \left(\frac{x^2}{1-x}\right)^k \\ &=\; \frac{1}{1-x^2/(1-x)} \;=\; \frac{1-x}{1-x-x^2} \;=\; 1+\frac{x^2}{1-x-x^2}. \end{aligned}$$

Here, $h_n = [x^n]H(x)$ is the number of compositions in $\mathcal{H}$ of size n.

Example 2.20. Let $\mathcal{J}$ be the set of all compositions in which each part is odd.

- The allowed sizes for a part are $P = \{1, 3, 5, ...\}$. The generating series for a single part is

$$\Phi_P(x) \;=\; \sum_{i=0}^{\infty} x^{2i+1} = x^1 + x^3 + x^5 + \cdots \;=\; \frac{x}{1-x^2}.$$

- The length can be any natural number $k \in \mathbb{N}$. By the Product Lemma, the generating series for compositions in $\mathcal{J}$ of length k is

$$\left(\frac{x}{1-x^2}\right)^k.$$

- Since $\mathcal{J} = P^*$, the String Lemma implies that

$$\begin{aligned} J(x) \;&=\; \Phi_{\mathcal{J}}(x) \;=\; \sum_{n=0}^{\infty} j_n\, x^n \;=\; \frac{1}{1-x/(1-x^2)} \\ &=\; \frac{1-x^2}{1-x-x^2} \;=\; 1+\frac{x}{1-x-x^2}. \end{aligned}$$

Here, $j_n = [x^n]J(x)$ is the number of compositions in $\mathcal{J}$ of size n.

Example 2.21. The sets $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{J}$ in Examples 2.18 to 2.20 have very similar generating series. In fact, after a little thought one sees that for all $n \geq 2$,

$$[x^n]\,H(x) \;=\; [x^{n-1}]\,J(x) \;=\; [x^{n-2}]\,F(x) \;=\; [x^{n-2}]\,\frac{1}{1-x-x^2}.$$

This means that for all $n \geq 2$, we have $h_n = j_{n-1} = f_{n-2}$, so for the sizes of sets we have $|\mathcal{H}_n| = |\mathcal{J}_{n-1}| = |\mathcal{F}_{n-2}|$. We have proven these equalities even though we don't yet know what those numbers actually are! This seems slightly magical, but it works.

Since these sets have the same sizes there must be bijections between them to explain this fact. Constructing such bijections is left to Exercise 2.17. As a starting point, here are the sets for $n = 7$:

$\mathcal{H}_7$	$\mathcal{J}_6$	$\mathcal{F}_5$
(7)	$(5,1)$	$(2,2,1)$
$(5,2)$	$(1,5)$	$(2,1,2)$
$(2,5)$	$(3,3)$	$(1,2,2)$
$(4,3)$	$(3,1,1,1)$	$(2,1,1,1)$
$(3,4)$	$(1,3,1,1)$	$(1,2,1,1)$
$(3,2,2)$	$(1,1,3,1)$	$(1,1,2,1)$
$(2,3,2)$	$(1,1,1,3)$	$(1,1,1,2)$
$(2,2,3)$	$(1,1,1,1,1,1)$	$(1,1,1,1,1)$

(It need not be the case that the bijections match up these sets of compositions line by line in this table.) In Section 4.1 we will determine the coefficients of the power series $1/(1-x-x^2)$, answering the counting problem for these sets $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{J}$.

Example 2.22. Let $\mathcal{Q}$ be the set of all compositions in which each part is at least two, and the number of parts is even.

- The allowed sizes for a part are $P = \{2, 3, 4, ...\}$. The generating series for a single part is

$$\Phi_P(x) \;=\; \sum_{c=2}^{\infty} x^c = x^2 + x^3 + x^4 + \cdots \;=\; \frac{x^2}{1-x}.$$

- The length is an even natural number $k = 2j$ for some $j \in \mathbb{N}$. By the Product Lemma, the generating series for a composition in $\mathcal{Q}$

of length $2j$ is

$$\left(\frac{x^2}{1-x}\right)^{2j}.$$

- Since $\mathcal{Q} = (P^2)^*$, the String Lemma implies that

$$\begin{aligned} Q(x) &= \Phi_{\mathcal{Q}}(x) = \sum_{n=0}^{\infty} q_n\, x^n = \sum_{j=0}^{\infty} \left(\frac{x^2}{1-x}\right)^{2j} \\ &= \frac{1}{1 - x^4/(1-x)^2} = \frac{(1-x)^2}{(1-x)^2 - x^4} \\ &= \frac{1-2x+x^2}{1-2x+x^2-x^4} = 1 + \frac{x^4}{1-2x+x^2-x^4}. \end{aligned}$$

Here $q_n = [x^n]Q(x) = |\mathcal{Q}_n|$ is the number of compositions in $\mathcal{Q}$ of size n. In Example 4.11 we will see how to calculate the first several values of $|\mathcal{Q}_n|$.

2.4 Subsets with Restrictions.

The theory above for compositions can be used to obtain generating series for subsets of natural numbers subject to some restrictions on the "gaps" between consecutive elements of the subset. This is because of the following correspondence between such subsets and nonempty compositions.

Proposition 2.23. *Let $\mathcal{U}$ be the set of pairs (n, A) in which $n \in \mathbb{N}$ is a natural number and $A \subseteq \{1, 2, ..., n\}$ is a subset. Let $\mathcal{C} \smallsetminus \{\varepsilon\}$ be the set of nonempty compositions. There is a bijection $\mathcal{U} \rightleftharpoons \mathcal{C} \smallsetminus \{\varepsilon\}$ between these two sets.*

Proof. We define mutually inverse bijections $\mathcal{U} \rightleftharpoons \mathcal{C} \smallsetminus \{\varepsilon\}$ as follows.

First, given (n, A) in $\mathcal{U}$, sort the elements of $A = \{a_1, a_2, ..., a_k\}$ into increasing order:

$$1 \leq a_1 < a_2 < \cdots < a_k \leq n.$$

For convenience, put $a_0 = 0$ and $a_{k+1} = n + 1$. Now define $c_i = a_i - a_{i-1}$ for all $1 \leq i \leq k+1$, and let $\gamma = (c_1, c_2, ..., c_{k+1})$. Notice that each c_i is a positive integer, so that γ is a composition of length $k+1$. Since $k+1$ is at least one, this composition γ is not empty, so that γ is in the set $\mathcal{C} \smallsetminus \{\varepsilon\}$. The size of this composition is

$$|\gamma| = \sum_{i=1}^{k+1} c_i = \sum_{i=1}^{k+1} (a_i - a_{i-1}) = a_{k+1} - a_0 = n + 1,$$

and its length is $\ell(\gamma) = |A| + 1$.

Conversely, given a nonempty composition $\gamma = (c_1, c_2, ..., c_\ell)$ in $\mathcal{C} \smallsetminus \{\varepsilon\}$, notice that $\ell \geq 1$. We define $a_j = c_1 + c_2 + \cdots + c_j$ for each $1 \leq j \leq \ell - 1$, and let $A = \{a_1, a_2, ..., a_{\ell-1}\}$ and $n = |\gamma| - 1$. This defines a pair (n, A) in the set $\mathcal{U}$.

One can check that these constructions give a pair of mutually inverse bijections $\mathcal{U} \rightleftharpoons \mathcal{C} \smallsetminus \{\varepsilon\}$ by using Proposition 1.11. This is left as Exercise 2.18 □

The bijection of Proposition 2.23 can be neatly summarized in a little table, as follows:

$$\begin{aligned} \mathcal{U} &\rightleftharpoons \mathcal{C} \smallsetminus \{\varepsilon\} \\ (n, A) &\leftrightarrow \gamma \\ n &= |\gamma| - 1 \\ |A| &= \ell(\gamma) - 1. \end{aligned}$$

Many variations on this bijection are possible, by putting some restrictions on the allowed "gaps" between elements of the subset A and then analyzing the corresponding set of (nonempty) compositions. When deriving these generating series, we have to be careful that the n in the pair (n, A) corresponds to $|\gamma| - 1$ for the corresponding composition γ.

Example 2.24. For each $n \in \mathbb{N}$, let r_n be the number of subsets of $\{1, ..., n\}$ that do not contain two consecutive numbers (like a and $a+1$). We obtain a formula for the generating series $R(x) = \sum_{n=0}^{\infty} r_n x^n$ using the ideas of Proposition 2.23.

For $n \in \mathbb{N}$, let $\mathcal{R}_n$ be the set of pairs (n, A) with A as in the statement of the problem, and let $\mathcal{R} = \bigcup_{n=0}^{\infty} \mathcal{R}_n$. Then $|\mathcal{R}_n| = r_n$ for all $n \in \mathbb{N}$, and we want to determine the generating series for the set $\mathcal{R}$ with respect to the weight function $\omega(n, A) = n$.

The first question to ask is: which nonempty compositions correspond to pairs in the set $\mathcal{R}$? Notice that (n, A) is in $\mathcal{R}$ if and only if the corresponding composition $\gamma = (c_1, c_2, ..., c_\ell)$ of size $n+1$ has $c_i \geq 2$ for all $2 \leq i \leq \ell - 1$. It is possible that maybe $c_1 = 1$ or $c_\ell = 1$, though. Let $\mathcal{M}$ be the set of compositions corresponding to pairs in $\mathcal{R}$. Notice that

$$M(x) = \Phi_{\mathcal{M}}(x) = \sum_{\gamma \in \mathcal{M}} x^{|\gamma|} = \sum_{(n,A) \in \mathcal{R}} x^{n+1} = x\, R(x).$$

The compositions in $\mathcal{M}$ can be described as follows.

- The first and last parts are positive integers.
- Parts other than the first and last parts are integers greater than or equal to 2.
- The length is at least one (since it is one more than the size of the corresponding subset).

A part that is a positive integer has generating series $x/(1-x)$, and a part that is at least 2 has generating series $x^2/(1-x)$, as we have seen. Now we do a case analysis by the number of parts, using the Product Lemma.

- For $\ell = 1$ part the generating series is $x/(1-x)$.
- For $\ell \geq 2$ parts the generating series is

$$\frac{x}{1-x}\left(\frac{x^2}{1-x}\right)^{\ell-2}\frac{x}{1-x} = \frac{x^{2\ell-2}}{(1-x)^\ell}.$$

Combining the contributions for all lengths $\ell \geq 1$ using the Sum Lemma, we have

$$\begin{aligned} xR(x) = M(x) &= \frac{x}{1-x} + \sum_{\ell=2}^{\infty} \frac{x^{2\ell-2}}{(1-x)^\ell} \\ &= \frac{x}{1-x} + \frac{x^2}{(1-x)^2}\sum_{j=0}^{\infty}\left(\frac{x^2}{1-x}\right)^j \end{aligned}$$

$$
\begin{aligned}
&= \frac{x}{1-x} + \frac{x^2}{(1-x)^2} \cdot \frac{1}{1-x^2/(1-x)} \\
&= \frac{x-x^2-x^3}{(1-x)(1-x-x^2)} + \frac{x^2}{(1-x)(1-x-x^2)} \\
&= \frac{x+x^2}{1-x-x^2}.
\end{aligned}
$$

It follows that

$$R(x) = \frac{1+x}{1-x-x^2}.$$

2.5 Proof of Inclusion/Exclusion.

In this section we prove Theorem 1.15, the Principle of Inclusion/Exclusion.

Lemma 2.25. *For any nonempty set T,*

$$\sum_{\varnothing \neq S \subseteq T} (-1)^{|S|-1} = 1.$$

Proof. Consider the identity

$$\sum_{S \subseteq \{1,2,\ldots,n\}} x^{|S|} \;=\; (1+x)^n$$

which was part of the proof of the Binomial Theorem. If T is any n-element set then

$$\sum_{S \subseteq T} x^{|S|} \;=\; (1+x)^n$$

as well (as can be seen by numbering the elements of T arbitrarily). Both sides are polynomials in x, so we can substitute $x=-1$. The result is

$$\sum_{S \subseteq T} (-1)^{|S|} \;=\; (1-1)^n \;=\; 0^n \;=\; 0,$$

because $n \geq 1$. (Note that $0^0 = 1$, since it is an empty product.) On the LHS we separate the term corresponding to $S = \varnothing$, and see that

$$1 + \sum_{\varnothing \neq S \subseteq T} (-1)^{|S|} = 0.$$

Rearranging this gives the desired formula. □

Recall the notation from Subsection 1.1.6: for any finite number of sets $A_1, A_2, \ldots, A_m$ and $\varnothing \neq S \subseteq \{1, 2, ..., m\}$, let

$$A_S = \bigcap_{i \in S} A_i.$$

So, for example, $A_{\{2,3,5\}} = A_2 \cap A_3 \cap A_5$.

Theorem 2.26 (Inclusion/Exclusion)**.** *Let $A_1, A_2, \ldots, A_m$ be finite sets. Then*

$$|A_1 \cup \cdots \cup A_m| = \sum_{\varnothing \neq S \subseteq \{1,2,\ldots,m\}} (-1)^{|S|-1} |A_S|.$$

Proof. Let $V = A_1 \cup \cdots \cup A_m$, and let $N_m = \{1, 2, .., m\}$. For each $v \in V$ let $T(v) = \{i \in N_m : \ v \in A_i\}$. Notice that $T(v) \neq \varnothing$, for all $v \in V$. Also notice that for $\varnothing \neq S \subseteq N_m$ we have $v \in A_S$ if and only if $\varnothing \neq S \subseteq T(v)$. Therefore, using Lemma 2.25 above, we have

$$\begin{aligned} \sum_{\varnothing \neq S \subseteq N_m} (-1)^{|S|-1} |A_S| &= \sum_{\varnothing \neq S \subseteq N_m} (-1)^{|S|-1} \sum_{v \in A_S} 1 \\ &= \sum_{v \in V} \sum_{\varnothing \neq S \subseteq T(v)} (-1)^{|S|-1} = \sum_{v \in V} 1 = |V|, \end{aligned}$$

as was to be shown. □

Example 2.27 (The Euler totient function)**.** For a positive integer n, the *Euler totient* of n is the number $\varphi(n)$ of integers b in the range $1 \leq b \leq n$ such that b and n are relatively prime. That is,

$$\varphi(n) = |\{b \in \{1, 2, ..., n\} : \ \gcd(b, n) = 1\}|.$$

We can use Inclusion/Exclusion to obtain a formula for $\varphi(n)$, as follows. Let the prime factorization of n be $n = p_1^{c_1} p_2^{c_2} \cdots p_m^{c_m}$, in which the p_i are pairwise distinct primes and the c_i are positive integers. For each $1 \leq i \leq m$, let

$$A_i := \{b \in N_n : \ p_i \text{ divides } b\}.$$

Then

$$\varphi(n) = |(N_n \smallsetminus (A_1 \cup \cdots \cup A_m))| = n - |A_1 \cup \cdots \cup A_m|.$$

Since the factors p_i are pairwise coprime, for any $\varnothing \neq S \subseteq N_m$ and $b \in N_n$ we have $b \in A_S$ if and only if $\prod_{i \in S} p_i$ divides b. Therefore,

$$|A_S| = \frac{n}{\prod_{i \in S} p_i}.$$

By Inclusion/Exclusion, it follows that

$$|A_1 \cup \cdots \cup A_m| = n \sum_{\varnothing \neq S \subseteq N_m} (-1)^{|S|-1} \prod_{i \in S} \frac{1}{p_i}.$$

Therefore

$$\begin{aligned} \varphi(n) &= n - n \sum_{\varnothing \neq S \subseteq N_m} (-1)^{|S|-1} \prod_{i \in S} \frac{1}{p_i} \\ &= n \sum_{S \subseteq N_m} (-1)^{|S|} \prod_{i \in S} \frac{1}{p_i} = n \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right). \end{aligned}$$

2.6 Exercises.

Exercise 2.1. Calculate the following coefficients.

(a) $[x^8](1-x)^{-7}$.
(b) $[x^{10}]x^6(1-2x)^{-5}$.
(c) $[x^8](x^3+5x^4)(1+3x)^6$.
(d) $[x^9]((1-4x)^5+(1-3x)^{-2})$.
(e) $[x^n](1-2tx)^{-k}$.
(f) $[x^{n+1}]x^k(1-4x)^{-2k}$.
(g) $[x^n]x^k(1-x^2)^{-m}$.
(h) $[x^n]((1-x^2)^{-k}+(1-7x3)^{-k})$.

Exercise 2.2. In each case, find an instance of a Binomial Series that begins as shown.

(a) $1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots$.
(b) $1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + \cdots$.
(c) $1 - x^3 + x^6 - x^9 + x^{12} - x^{15} + \cdots$.
(d) $1 + 2x^2 + 4x^4 + 8x^6 + 16x^8 + 32x^{10} + \cdots$.
(e) $1 - 4x^2 + 12x^4 - 32x^6 + 80x^8 - 192x^{10} + \cdots$.
(f) $1 + 6x + 24x^2 + 80x^3 + 240x^4 + 672x^5 + \cdots$.

Exercise 2.3. Give algebraic proofs of these identities from Exercise 1.7.

(a) For all $n \in \mathbb{N}$, $\sum_{k=0}^{n} \binom{n}{k} k = n\, 2^{n-1}$.
(b) For all $n \in \mathbb{N}$, $\sum_{k=0}^{n} \binom{n}{k} k(k-1) = n(n-1)2^{n-2}$.

Exercise 2.4. Calculate $[x^n](1+x)^{-2}(1-2x)^{-2}$. Give the simplest expression you can find.

Exercise 2.5.

(a) Let $a \geq 1$ be an integer. For each $n \in \mathbb{N}$, extract the coefficient of x^n from both sides of this power series identity:

$$\frac{(1+x)^a}{(1-x^2)^a} = \frac{1}{(1-x)^a}$$

to show that

$$\binom{n+a-1}{a-1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{a}{n-2k} \binom{k+a-1}{a-1}.$$

(b) Can you think of a combinatorial proof?

Exercise 2.6. Prove the Infinite Sum Lemma 2.11.

Exercise 2.7. Extend the Product Lemma 2.12 to the product of finitely many sets with weight functions.

Exercise 2.8. Show that for $m, n, k \in \mathbb{N}$,

$$\sum_{j=0}^{k} (-1)^j \binom{n+j-1}{j} \binom{m}{k-j} = \binom{m-n}{k}.$$

Exercise 2.9.

(a) Make a list of all the four-letter "words" that can be formed from the "alphabet" $\{a, b\}$. Define the weight of a word to be the number of occurrences of ab in it. Determine how many words there are of weight 0, 1 and 2. Determine the generating series.

(b) Do the same for five-letter words over the same alphabet, but, preferably, without listing all the words separately.

(c) Do the same for six-letter words.

Exercise 2.10.

(a) Consider throwing two six–sided dice, one red and one green. The weight of a throw is the total number of pips showing on the top faces of both dice (that is, the usual score). Make a table showing the number of throws of each weight, and write down the generating series.

(b) Do the same as for part (a), but throwing three dice: one red, one green, and one white.

Exercise 2.11. Construct a table, as in Exercise 2.10(a), if the weight of a throw is defined to be the absolute value of the difference between the numberof pips showing on the two dice. Also, write down the generating series.

Exercise 2.12. Let $\mathcal{S}$ be the set of ordered pairs (a, b) of integers with $0 \leq |b| \leq a$. Each part gives a function ω defined on the set $\mathcal{S}$. Determine whether or not ω is a weight function on the set $\mathcal{S}$. If it is not, then explain why not. If it is a weight function, then determine the generating series $\Phi_{\mathcal{S}}(x)$ of $\mathcal{S}$ with respect to ω, and write it as a polynomial or a quotient of polynomials.

(a) For (a, b) in $\mathcal{S}$, let $\omega((a, b)) = a$.
(b) For (a, b) in $\mathcal{S}$, let $\omega((a, b)) = a + b$.
(c) For (a, b) in $\mathcal{S}$, let $\omega((a, b)) = 2a + b$.

Exercise 2.13. Let $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}^4$ be the set of outcomes when rolling four six-sided dice. For $(a, b, c, d) \in \mathcal{S}$, define its weight to be $\omega(a, b, c, d) = a + b + c + d$. Consider the generating series $\Phi_{\mathcal{S}}(x)$ of $\mathcal{S}$ with respect to ω.

(a) Explain why $\Phi_{\mathcal{S}}(x) = \left(\dfrac{x - x^7}{1 - x}\right)^4$.
(b) How many outcomes in $\mathcal{S}$ have weight 19?
(c) Let m, d, k be positive integers. When rolling m dice, each of which has exactly d sides (numbered with $1, 2, ..., d$ pips, respectively), how many different ways are there to roll a total of k pips on the top faces of the dice? (Part (b) is the case $m = 4$, $d = 6$, $k = 19$.)

Exercise 2.14. Let $\mathcal{A}$ be a set with weight function $\omega : \mathcal{A} \to \mathbb{N}$. Show that for any $n \in \mathbb{N}$, the number of $\alpha \in \mathcal{A}$ with $\omega(\alpha) \leq n$ is

$$[x^n]\, \frac{1}{1 - x}\, \Phi_{\mathcal{A}}(x).$$

Exercise 2.15. For each of the following sets of compositions, obtain a rational function formula for the generating series of that set (with respect to size).

(a) Let $\mathcal{A}$ be the set of compositions of length congruent to 1 (modulo 3).
(b) Let $\mathcal{B}$ be the set of compositions of length congruent to 2 (modulo 3).
(c) Let $\mathcal{C}$ be the set of compositions of even length, with each part being at most 3.
(d) Let $\mathcal{D}$ be the set of compositions of odd length, with each part being at least 2.
(e) Let $\mathcal{E}$ be the set of compositions $\gamma = (c_1, c_2, ..., c_k)$ of any length, in which each part c_i is congruent to i (modulo 2). So c_1 is odd, c_2 is even, c_3 is odd, and so on. (Note that the empty composition $\varepsilon = ()$ is in the set $\mathcal{E}$.)

Exercise 2.16.

(a) Let $\mathcal{A}_n$ be the set of all compositions of size n in which every part is at most three. Obtain a formula for the generating series $\sum_{n=0}^{\infty} |\mathcal{A}_n| \, x^n$.
(b) Let $\mathcal{B}_n$ be the set of all compositions of size n in which every part is a positive integer that is not divisible by three. Obtain a formula for the generating series $\sum_{n=0}^{\infty} |\mathcal{B}_n| \, x^n$.
(c) Deduce that for all $n \geq 3$, $|\mathcal{B}_n| = |\mathcal{A}_n| - |\mathcal{A}_{n-3}|$.
(d)* Can you find a combinatorial proof of part (c)?

Exercise 2.17. Find bijections to explain the equalities in Example 2.21.

Exercise 2.18. Prove that the constructions in the proof of Proposition 2.23 define mutually inverse bijections $\mathcal{U} \rightleftharpoons \mathcal{C} \smallsetminus \{\varepsilon\}$.

Exercise 2.19. For each part, determine the generating series for the number of subsets S of $\{1, 2, ..., n\}$ subject to the stated restriction.

(a) Consecutive elements of S differ by at most 2.
(b) Consecutive elements of S differ by at least 3.
(c) Consecutive elements of S differ by at most 3.
(d) Consecutive elements of S differ by a number congruent to 1 (modulo 3).
(e) Consecutive elements of S differ by a number congruent to 2 (modulo 3).
(f) If $S = \{a_1 < a_2 < \cdots < a_k\}$ then $a_i \equiv i \pmod 2$ for all $1 \leq i \leq k$.
(g) Fix integers $1 \leq g < h$. If $S = \{a_1 < a_2 < \cdots < a_k\}$ then $g \leq a_i - a_{i-1} \leq h$ for $2 \leq i \leq k$.

Exercise 2.20. Fix $n, k \in \mathbb{N}$. Let $R(n, k)$ be the number of k-element subsets $S = \{a_1 < a_2 < \cdots < a_k\}$ of $\{1, 2, ..., n\}$ such that $a_i - a_{i-1} \geq i$ for all $2 \leq i \leq k$. Show that

$$R(n, k) = \binom{n - k(k-1)/2}{k}.$$

Exercise 2.21. Let $p(n)$ be a polynomial function of n.

(a) Prove, by induction on $d = \deg(p)$, that $p(n)$ can be written as a linear combination of $\binom{n+j}{j}$ for $j = 0, 1, 2, ..., d$.
(b) Briefly explain why there is a polynomial $A_p(x)$ such that

$$\sum_{n=0}^{\infty} p(n) x^n = \frac{A_p(x)}{(1-x)^{d+1}}.$$

(c) What can you say about the degree of $A_p(x)$? What can you say about the value of $A_p(1)$?

Chapter 3

Binary Strings.

This chapter presents a wide variety of examples to which the generating series technique of Chapter 2 applies. In Chapter 4 we will see how to use these generating series to answer the counting problems which arise. Similar calculations which provide even more information are presented in Chapter **??**.

Definition 3.1. A *binary string* is a finite sequence $\sigma = b_1 b_2 \cdots b_n$ in which each *bit* b_i is either 0 or 1. The number of bits is the *length* of the string, denoted $\ell(\sigma) = n$. Thus, a binary string of length n is an element of the Cartesian power $\{0,1\}^n$. A binary string of arbitrary length is an element of the set $\{0,1\}^* = \bigcup_{n=0}^{\infty} \{0,1\}^n$. There is exactly one binary string $\varepsilon = ()$ of length zero, the empty string with no bits.

Clearly, there are 2^n binary strings of length n, so that the generating series for binary strings with respect to length is

$$\Phi_{\{0,1\}^*}(x) = \sum_{\sigma \in \{0,1\}^*} x^{\ell(\sigma)} = \sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}.$$

We will see how to describe various subsets of binary strings in a way which allows us to determine their generating series (with respect to length).

3.1 Regular Expressions and Rational Languages.

Definition 3.2 (Regular Expressions.)**.** A *regular expression* is defined recursively, as follows.

- All of ε, 0, and 1 are regular expressions.
- If R and S are regular expressions, then so is R$\smile$S.
- If R and S are regular expressions, then so is RS.
  For any finite $k \in \mathbb{N}$ we also use R^k for the k-fold concatenation of R: that is $\mathsf{R}^2 = \mathsf{RR}$ and $\mathsf{R}^3 = \mathsf{RRR}$, and so on.
- If R is a regular expression, then so is R^*.

For example, $(\varepsilon \smile 0^*00)\,(1^*\,0^*00)^*\,1^*$ is a regular expression. These regular expressions are just formal syntactic constructions with no intrinsic meaning. However, we will interpret them in two different ways.

- A regular expression R will *produce* a subset $\mathcal{R} \subseteq \{0,1\}^*$. Such a subset is called a *rational language*. (See Definition 3.5.)
- A regular expression R will *lead to* a rational function $R(x)$. (See Definition 3.11.)

In general, the rational function $R(x)$ is quite meaningless. However, under favourable conditions on the expression R, it turns out that $R(x) = \Phi_{\mathcal{R}}(x)$ is the generating series of the set $\mathcal{R}$ with respect to length. Then the machinery of Chapter 4 can be applied.

Definition 3.3 (Concatenation Product)**.** Let $\alpha, \beta \in \{0,1\}^*$ be binary strings – so $\alpha = a_1a_2\cdots a_m$ and $\beta = b_1b_2\cdots b_n$. The *concatenation* of α and β is the string

$$\alpha\beta = a_1a_2\cdots a_mb_1b_2\cdots b_n.$$

Let $\mathcal{A}, \mathcal{B} \subseteq \{0,1\}^*$ be sets of binary strings. The *concatenation product* $\mathcal{AB}$ is the set

$$\mathcal{AB} = \{\alpha\beta :\ \alpha \in \mathcal{A} \text{ and } \beta \in \mathcal{B}\}.$$

Example 3.4. Consider the sets $\mathcal{A} = \{011, 01\}$ and $\mathcal{B} = \{101, 1101\}$. There are four ways to concatenate a string in $\mathcal{A}$ followed by a string in $\mathcal{B}$:

$$011.101, \ 011.1101, \ 01.101, \ 01.1101.$$

Here, the dot . indicates the point at which the concatenation takes place. However, this information is not recorded when passing from $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ to their concatenation $\alpha\beta$. Thus the concatenation product $\mathcal{AB}$ consists of the strings

$$011101, \ 0111101, \ 01101, \ 011101.$$

The string 011101 is produced twice. The concatenation product $\mathcal{AB}$ has only three elements:

$$\mathcal{AB} = \{011101, 0111101, 01101\}.$$

Definition 3.5 (Rational Languages.)**.** A *rational language* is a set $\mathcal{R} \subseteq \{0, 1\}^*$ of binary strings that is *produced* by a regular expression; this is defined recursively as follows.

- To begin with, ε produces $\{\varepsilon\}$ and 0 produces $\{0\}$ and 1 produces $\{1\}$.
- If R produces $\mathcal{R}$ and S produces $\mathcal{S}$, then R$\smile$S produces $\mathcal{R} \cup \mathcal{S}$.
- If R produces $\mathcal{R}$ and S produces $\mathcal{S}$, then RS produces $\mathcal{RS}$.
- If R produces $\mathcal{R}$ then R* produces $\mathcal{R}^* = \bigcup_{k=0}^{\infty} \mathcal{R}^k$. Here $\mathcal{R}^k$ is the concatenation product of k copies of $\mathcal{R}$.

It is important to note that a rational language can be produced by many different regular expressions, as we will see.

In Definition 3.5, when R produces $\mathcal{R}$ and S produces $\mathcal{S}$, it might happen that $\mathcal{R} \cup \mathcal{S}$ is not a disjoint union of sets. Also, the concatenation product $\mathcal{RS}$ is not the same as the Cartesian product $\mathcal{R} \times \mathcal{S}$, as Example 3.4 shows. These facts lead to complications which are addressed in Section 3.2.

Example 3.6. Here are some easy examples.

- The regular expression 1^* produces the rational language
$$\{1\}^* = \{\varepsilon, 1, 11, 111, 1111, ...\}$$
of all finite strings of 1s (including the empty string ε).
- The regular expression $(1 \smile 11)^*$ also produces the set $\{1\}^*$.
- The regular expression $1(11)^*$ produces the rational language of all strings of 1s of odd (positive) length:
$$\{1, 111, 11111, ...\}.$$
- The regular expression $(0 \smile 1)^*$ produces the rational language $\{0, 1\}^*$ of all binary strings.
- The regular expression $1^*(01^*)^*$ also produces the rational language $\{0, 1\}^*$ of all binary strings.

Example 3.7. Not every set of binary strings is a rational language.

- The regular expression $(01)^*$ produces the rational language
$$\{\varepsilon, 01, 0101, 010101, 01010101, ...\}.$$
For every even natural number $2j$ there is exactly one string of length $2j$ in this set.
- The set
$$\{\varepsilon, 01, 0011, 000111, 00001111, ...\}$$
is not a rational language, even though for every even natural number $2j$ there is exactly one string of length $2j$ in this set.

The problem with Example 3.7 is that to describe the second set we would need an expression like $\bigcup_{j=0}^{\infty} 0^j 1^j$. However, an infinite union like this is not allowed according to Definition 3.2. The underlying difficulty is that a regular expression has a "finite memory" and cannot remember arbitrarily large numbers, like the $j \in \mathbb{N}$ needed in the above expression. There is a close connection between rational languages and finite state machines, and this is a central topic in the theory of computation.

3.2 Unambiguous Expressions.

Definition 3.8 (Unambiguous Expression)**.** Let R be a regular expression that produces a rational language $\mathcal{R}$. Then R is *unambiguous* if every string in $\mathcal{R}$ is produced exactly once by R. If an expression is not unambiguous then it is *ambiguous*.

As is usual with regular expressions, whether or not it is unambiguous can be decided recursively.

Lemma 3.9 (Unambiguous Expression)**.** *Let R and S be unambiguous expressions producing the sets $\mathcal{R}$ and $\mathcal{S}$, respectively.*

- *The expressions ε and 0 and 1 are unambiguous.*
- *The expression $\mathsf{R} \smile \mathsf{S}$ is unambiguous if and only if $\mathcal{R} \cap \mathcal{S} = \varnothing$, so that $\mathcal{R} \cup \mathcal{S}$ is a disjoint union of sets.*
- *The expression RS is unambiguous if and only if there is a bijection $\mathcal{RS} \rightleftharpoons \mathcal{R} \times \mathcal{S}$ between the concatenation product $\mathcal{RS}$ and the Cartesian product $\mathcal{R} \times \mathcal{S}$. In other words, for every string $\alpha \in \mathcal{RS}$ there is exactly one way to write $\alpha = \rho\sigma$ with $\rho \in \mathcal{R}$ and $\sigma \in \mathcal{S}$.*
- *The expression R^* is unambiguous if and only if each of the concatenation products R^k is unambiguous and the union $\bigcup_{k=0}^{\infty} \mathcal{R}^k$ is a disjoint union of sets.*

Proof. Exercise 3.1. □

Example 3.10. Here are some easy examples.

- The expression 1^* is unambiguous.
- The expression $(1 \smile 11)^*$ is ambiguous: $1.11 = 11.1 = 1.1.1$.
- The expression $(0 \smile 1)^*$ is unambiguous. This expression produces each string in $\{0, 1\}^*$ one bit at a time, so each string is produced in exactly one way.
- The expression $1^*(01^*)^*$ is also unambiguous. First, the expression 01^* is unambiguous, since it produces a 0 followed by a (possibly empty) string of 1s – each such string is produced exactly once.

Now $(01^*)^k$ is unambiguous for any $k \in \mathbb{N}$, since any string it produces will begin with a 0, it will have k bits equal to 0, and the strings of 1s following these 0s can only be constructed in one way. Next, $(01^*)^*$ is unambiguous since for each $k \in \mathbb{N}$, the strings produced by $(01^*)^k$ have exactly k bits equal to 0 (so the union of sets corresponding to the outer $*$ is a disjoint union). Finally, $1^*(01^*)^*$ is unambiguous since for any string it produces, the length of the initial string of 1s is also determined uniquely.

3.2.1 Translation into generating series.

We translate regular expressions into rational functions as follows.

Definition 3.11. A regular expression *leads to* a rational function; this is defined recursively, as follows. Assume that R and S are regular expressions that lead to $R(x)$ and $S(x)$, respectively.

- To begin with, ε leads to 1 and 0 leads to x and 1 leads to x.
- The expression $\mathsf{R} \smile \mathsf{S}$ leads to $R(x) + S(x)$.
- The expression RS leads to $R(x) \cdot S(x)$.
- The expression R^* leads to $1/(1 - R(x))$.

It is easy to see (again recursively!) that if R leads to $R(x)$, then $R(x)$ is a rational function.

Example 3.12. Here are some easy examples.

- The unambiguous expression 1^* leads to $1/(1-x)$.
- The ambiguous expression $(1 \smile 11)^*$ leads to $1/(1-(x+x^2))$. But this expression produces the same rational language as 1^*.
- The unambiguous expression $(0 \smile 1)^*$ leads to

$$\frac{1}{1-(x+x)} = \frac{1}{1-2x}.$$

- The unambiguous expression $1^*(01^*)^*$ leads to

$$\frac{1}{1-x} \cdot \frac{1}{1-x \cdot 1/(1-x)} = \frac{1}{1-2x}.$$

Theorem 3.13. *Let* R *be a regular expression producing the rational language* $\mathcal{R}$ *and leading to the rational function* $R(x)$*. If* R *is an unambiguous expression for* $\mathcal{R}$ *then* $R(x) = \Phi_{\mathcal{R}}(x)$*, the generating series for* $\mathcal{R}$ *with respect to length.*

Sketch of proof. The proof of this is, as usual, recursive. Or, one could say it goes by induction on the complexity of the expression R, and uses the fact that R is unambiguous. Certainly, each of ε, 0, and 1 are unambiguous and lead to the correct generating series for the sets $\{\varepsilon\}$, $\{0\}$, and $\{1\}$, respectively. The induction step follows from Lemma 3.9 and the Sum, Product, and String Lemmas of Subsection 2.2.2, because each of the operations is unambiguous. □

Example 3.14. If the regular expression R producing $\mathcal{R}$ is ambiguous, then the rational function $R(x)$ is in general meaningless.

- For example, consider the regular expression $(\varepsilon \smile 1)^*$, which produces the rational language $\{1\}^*$. It is an ambiguous expression. In fact, it produces every string of 1s in infinitely many ways. The generating series for the set $\{1\}^*$ is $1/(1-x)$. The expression $(\varepsilon \smile 1)^*$ leads to $1/(1-(1+x)) = -x^{-1}$. If this were a generating series it would say that there are exactly -1 objects of size -1, and nothing else. This makes no sense.
- Similarly, the ambiguous expression $(1 \smile 11)^*$ also produces the set $\{1\}^*$. However, the expression $(1 \smile 11)^*$ leads to $1/(1-x-x^2)$, which is also incorrect.

3.2.2 Block decompositions.

Definition 3.15 (Blocks of a string)**.** Let $\sigma = b_1 b_2 b_3 \cdots b_n$ be a binary string of length n. A *block* of σ is a nonempty maximal subsequence of consecutive equal bits. To be clearer, a block is a nonempty subsequence $b_i b_{i+1} \cdots b_j$ of consecutive bits all of which are the same (all are 0, or all are 1), and which cannot be made longer. So, either $i = 1$ or $b_{i-1} \neq b_i$, and either $j = n$ or $b_{j+1} \neq b_j$.

Example 3.16. The blocks of the string $1100010111001000111100 1011$ are separated by dots here:

$$11.000.1.0.111.00.1.000.1111.00.1.0.11$$

Proposition 3.17 (Block Decompositions.)**.** *The regular expressions*

$$0^*(1^*10^*0)^*1^* \quad \textit{and} \quad 1^*(0^*01^*1)^*0^*$$

are unambiguous expressions for the set $\{0,1\}^*$ *of all binary strings. They produce each binary string block by block.*

Sketch of proof. By symmetry it is enough to consider the first expression. The middle part 1^*10^*0 produces a block of 1s followed by a block of 0s. This concatenation is unambiguous. The repetition of this $(1^*10^*0)^*$ is also unambiguous, since each pass through the repetition starts with a 1 and ends with a 0. (Try it out on Example 3.16.) But the string we want to build might start with a block of 0s: the initial 0^* allows this but does not require it, since $0^* = \varepsilon \smile 0^*0$. The final 1^* similarly allows the string to end with a block of 1s, but does not require it. All the operations are unambiguous, so the whole expression is unambiguous. □

Example 3.18. The block decomposition $0^*(1^*10^*0)^*1^*$ is unambiguous, and produces $\{0,1\}^*$. It had better lead to the right generating series! After a bit of calculation, we see that it leads to

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2/(1-x)^2} \cdot \frac{1}{1-x} = \frac{1}{1-2x},$$

which is good.

Example 3.19. Let $\mathcal{G}$ be the set of binary strings in which every block of 1s has odd length. What is the generating series for $\mathcal{G}$ with respect to length? We will modify the block decomposition $0^*(1^*10^*0)^*1^*$ for all binary strings. The expression 1^*1 in the middle produces a block of 1s. The expression $1^* = \varepsilon \smile 1^*1$ produces either the empty string or a block of 1s. If we want a block of 1s of odd length, then that is produced by $(11)^*1$. So the expression

$$\mathsf{G} = 0^* \, ((11)^* 1 \, 0^* 0)^* \, (\varepsilon \smile (11)^* 1)$$

is a block decomposition for the set $\mathcal{G}$ in question. It is therefore an unambiguous expression that produces $\mathcal{G}$. This expression leads to

$$\begin{aligned} G(x) &= \frac{1}{1-x} \cdot \frac{1}{1-(x/(1-x^2))(x/(1-x))} \cdot \left(1 + \frac{x}{1-x^2}\right) \\ &= \frac{1+x-x^2}{(1-x)(1-x^2)-x^2} = \frac{1+x-x^2}{1-x-2x^2+x^3}. \end{aligned}$$

By Theorem 3.13, $G(x) = \Phi_{\mathcal{G}}(x)$ is the generating series of the set $\mathcal{G}$. It follows from Theorem 4.8 that the number g_n of strings of length n in $\mathcal{G}$ satisfies the linear recurrence relation with initial conditions given by

$$g_n - g_{n-1} - 2g_{n-2} + g_{n-3} = \begin{cases} 1 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ -1 & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases}$$

(with the convention that $g_n = 0$ for all $n < 0$). This gives the initial conditions $g_0 = 1$, $g_1 = 2$, $g_2 = 3$ (which can be checked directly), and the recurrence $g_n = g_{n-1} + 2g_{n-2} - g_{n-3}$ for all $n \geq 3$. It is easy to calculate the first several of these numbers.

n	0	1	2	3	4	5	6	7	8
g_n	1	2	3	6	10	19	33	61	108

To get an exact formula for g_n we would need to factor the denominator $1 - x - 2x^2 + x^3$ to find its inverse roots. They turn out to be slightly horrible complex numbers, so we won't do it.

Example 3.20. Let $\mathcal{H}$ be the set of binary strings in which each block of 0s has length one. It is not hard to see that $(\varepsilon \smile 0)\,(1^*1\,0)^*\,1^*$ is a block decomposition for $\mathcal{H}$, and is therefore unambiguous. This expression leads to the formula

$$H(x) = \sum_{n=0}^{\infty} h_n x^n = (1+x) \cdot \frac{1}{1 - x^2/(1-x)} \cdot \frac{1}{1-x} = \frac{1+x}{1-x-x^2}.$$

This resembles Examples 2.21 and 2.24 yet again! These are the Fibonacci numbers – see Example 4.1. With $1/(1-x-x^2) = \sum_{n=0}^{\infty} f_n x^n$, we see that for $n \geq 1$,

$$h_n = [x^n]H(x) = f_n + f_{n-1} = f_{n+1}.$$

Let's reality check this for $n = 5$. The strings in $\mathcal{H}_5$ are 11111, five strings with one 0, six strings with two 0s, and 01010. And indeed, $f_6 = 13$. Neat!

3.2.3 Prefix decompositions.

Example 3.21. Consider the regular expression $(0^*1)^*0^*$. We claim that this is unambiguous. To see this, let $\sigma = b_1 b_2 \cdots b_n$ be a binary string produced by this expression. As an example, take

$$0011101011000101100 0.$$

How can this string be produced? The repetition $(0^*1)^*$ produces a binary string by chopping it into pieces after each occurrence of the bit 1. But any string produced by this expression is either empty or ends with a 1. The final "suffix" 0^* in the expression allows the possibility that the string might end with some 0s. The string above is produced as

$$001.1.1.01.01.1.0001.01.1.000$$

and this is the only way it is produced. This rule – "chop the string into pieces after each occurrence of the bit 1" – gives a unique way to produce each binary string from the expression $(0^*1)^*0^*$. It follows that $(0^*1)^*0^*$ is an unambiguous expression for the set $\{0,1\}^*$ of all binary strings. And, indeed, $(0^*1)^*0^*$ leads to

$$\frac{1}{1-x/(1-x)} \cdot \frac{1}{1-x} = \frac{1}{1-2x}.$$

In general, a *prefix decomposition* for a set of binary strings is a regular expression of the form $\mathsf{A}^*\mathsf{B}$. The idea is to chop each string that is produced into initial segments produced by the expression A. There might be a terminal segment produced by the expression B. (It is possible that $\mathsf{B} = \varepsilon$.) One does have to argue that such expressions are unambiguous, but – as in Example 3.21 – this can usually be done by checking that:

- there is at most one way for a binary string to begin with an initial segment produced by A, and
- if the string does not begin with an initial segment produced by A then it is produced by B.

Both of the expressions A and B must be unambiguous, too, of course.

Similarly, a *postfix decomposition* has the form $\mathsf{A}(\mathsf{B}^*)$.

3.3 Recursive Decompositions.

Recursive decompositions are more general than regular expressions, and can produce sets of binary strings that are more general than rational languages. The added feature is that in addition to the elementary building blocks "ε" and "0" and "1" in a regular expression, we are also allowed to use letters that stand for sub-expressions on both sides of an equation.

Example 3.22. A simple example is the expression $\mathsf{S} = \varepsilon \smile (0 \smile 1)\mathsf{S}$. Since S appears on both the LHS and the RHS, in some sense this "defines S in terms of itself". This is the key idea behind recursive expressions. One reads this expression as saying that a string produced by S is either empty, or it consists of a single bit (either 0 or 1) followed by another (shorter) string produced by S. By induction on the length of the string, one sees that every binary string is produced exactly once in this way.

Translating the above expression into rational functions by analogy with Definition 3.11 leads to an equation for $S(x)$:

$$S(x) = 1 + (x + x)S(x).$$

This is easily solved to yield $S(x) = 1/(1 - 2x)$.

Example 3.23. As in Example 3.7, consider the set of strings

$$\mathcal{B} = \{\varepsilon, 01, 0011, 000111, ...\}.$$

This is not a rational language, but it can be described by the unambiguous recursive expression $\mathsf{B} = \varepsilon \smile 0\,\mathsf{B}\,1$. This leads to the equation $B(x) = 1 + x^2 B(x)$, and thence to $B(x) = 1/(1 - x^2)$.

Subsection 4.4.2 presents a more substantial example of a recursive decomposition which produces a set of binary strings that is not a rational language. The generating series for that example is not even a rational function.

3.3.1 Excluded substrings.

Let $\kappa \in \{0,1\}^*$ be a nonempty binary string. We say that $\sigma \in \{0,1\}^*$ *contains* κ if there are (possibly empty) binary strings α, β such that $\sigma = \alpha\,\kappa\,\beta$. If σ does not contain κ then σ *avoids* or *excludes* κ. Let $\mathcal{A}_\kappa \subset \{0,1\}^*$ be set of strings that avoid κ. We will develop a general method for calculating the generating series $A_\kappa(x)$.

Example 3.24. As an easy first example, consider the case $\kappa = 01011$. Let $\mathcal{A}$ be the set of strings avoiding 01011, and let $\mathcal{B}$ be the set of strings that have exactly one occurrence of 01011, at the very end (that is, as a suffix). Consider the strings in $\mathcal{A} \cup \mathcal{B}$. Such a string is either empty, or it ends with either a 0 or a 1. If this string is not empty, then removing the last bit leaves a (possibly empty) string in $\mathcal{A}$ (because of the way the sets $\mathcal{A}$ and $\mathcal{B}$ are defined). This translates into the relation

$$\mathsf{A} \smile \mathsf{B} \;=\; \varepsilon \smile \mathsf{A}\,(0 \smile 1)$$

for expressions A and B producing $\mathcal{A}$ and $\mathcal{B}$, respectively. This leads to the equation

$$A(x) + B(x) = 1 + 2x\,A(x)$$

for the generating series.

We need another equation in order to determine $A(x)$ and $B(x)$. If $\beta = \alpha\,01011$ is an arbitrary string in $\mathcal{B}$, then α is in $\mathcal{A}$, again from the way the sets $\mathcal{A}$ and $\mathcal{B}$ are defined. We claim that the converse also holds: if $\alpha \in \mathcal{A}$ then $\beta = \alpha\,01011$ is in $\mathcal{B}$. To see this we need to show that the only occurrence of 01011 as a substring of β is the one at the very end. We know that $\alpha \in \mathcal{A}$ does not contain 01011 as a substring. So if there is another occurrence of 01011 inside β then it must "overlap" the final 01011 in at least one position (but not in all positions). For this particular excluded substring this is not possible, as the following table shows:

$$\begin{array}{cccc|ccccc}
. & . & . & . & 0 & 1 & 0 & 1 & 1 \\
\hline
0 & 1 & 0 & 1 & \underline{1} & . & . & . & . \\
. & 0 & 1 & 0 & \underline{1} & 1 & . & . & . \\
. & . & 0 & 1 & 0 & 1 & \underline{1} & . & . \\
. & . & . & 0 & \underline{1} & \underline{0} & \underline{1} & 1 & .
\end{array}$$

In each row after the first, there is at least one position at which the shifted 01011 disagrees with the substring in the first row. So 01011 cannot overlap itself in a nontrivial way. This gives the relation $\mathsf{B} = \mathsf{A}\,01011$, yielding the equation $B(x) = x^5 A(x)$. Substituting this into the first equation gives $1 + 2x\,A(x) = (1 + x^5)\,A(x)$, which is easily solved. We conclude that $A(x) = 1/(1 - 2x + x^5)$.

Example 3.25. As a slightly trickier example, consider the case $\kappa = 01101$. Again, let $\mathcal{A}$ be the set of strings avoiding 01101, and let $\mathcal{B}$ be the set of strings that have exactly one occurrence of 01101, at the very end (that is, as a suffix). As in the previous example, a string in $\mathcal{A} \cup \mathcal{B}$ is either empty, or it ends with either a 0 or a 1. The reasoning of the previous example also holds in this case, translating into the same relation

$$\mathsf{A} \smile \mathsf{B} = \varepsilon \smile \mathsf{A}\,(0 \smile 1)$$

and the same equation $A(x) + B(x) = 1 + 2x\,A(x)$ for the generating series.

The second equation is the slightly trickier part, because the string 01101 can overlap itself in a nontrivial way. As in the previous example, we still have the set inclusion $\mathcal{B} \subseteq \mathcal{A}\,01101$. But the reverse inclusion does not hold in this case: for example $011.01101 = 01101.101$ is in $\mathcal{A}\,01101$ but not in $\mathcal{B}$, because it contains a substring 01101 that is not at the very end.

$$\begin{array}{cccc|ccccc}
. & . & . & . & 0 & 1 & 1 & 0 & 1 \\
\hline
0 & 1 & 1 & 0 & \underline{1} & . & . & . & . \\
. & 0 & 1 & 1 & 0 & 1 & . & . & . \\
. & . & 0 & 1 & \underline{1} & \underline{0} & 1 & . & . \\
. & . & . & 0 & \underline{1} & 1 & \underline{0} & \underline{1} & .
\end{array}$$

Looking at all the possible ways that 01101 can overlap itself, we see that in this case $\mathcal{A}\,01101 = \mathcal{B} \cup \mathcal{B}\,101$. This gives $\mathsf{A}\,01101 = \mathsf{B}\,(\varepsilon \smile 101)$ and $x^5 A(x) = (1+x^3)B(x)$. Substituting $B(x) = x^5 A(x)/(1+x^3)$ into the first equation and solving for $A(x)$ yields

$$A(x) = \frac{1+3x+2x^2}{1-2x-2x^2+x^3+x^5}.$$

The proof of the general result follows exactly the same pattern.

Theorem 3.26. *Let $\kappa \in \{0,1\}^*$ be a nonempty string of length n, and let $\mathcal{A} = \mathcal{A}_\kappa$ be the set of binary strings that avoid κ. Let $\mathcal{C}$ be the set of all nonempty suffixes γ of κ such that $\kappa\,\gamma = \eta\,\kappa$ for some nonempty prefix η of κ. Let $C(x) = \sum_{\gamma\in\mathcal{C}} x^{\ell(\gamma)}$. Then*

$$A(x) = \frac{1+C(x)}{(1-2x)(1+C(x))+x^n}.$$

Proof. Let $\mathcal{B}$ be the set of strings that contain exactly one occurrence of κ, at the very end. Any string in $\mathcal{A} \cup \mathcal{B}$ is either empty or in the set $\mathcal{A}\{0,1\}$, so that

$$\mathcal{A} \cup \mathcal{B} = \{\varepsilon\} \cup \mathcal{A}\,\{0,1\}.$$

This yields $A(x) + B(x) = 1 + 2x\,A(x)$.

The key observation is that if $\beta \in \mathcal{B}$ and ρ is a proper prefix of β then $\rho \in \mathcal{A}$. Consequently, $\mathcal{B} \subseteq \mathcal{A}\,\kappa$. Conversely, consider any $\sigma = \alpha\,\kappa$ in $\mathcal{A}\,\kappa$.

Maybe $\sigma \in \mathcal{B}$. If $\sigma \notin \mathcal{B}$, then there is an "early" occurrence of κ in $\alpha\,\kappa$ that is not the occurrence at the very end. All such early occurrences of κ must overlap the final occurrence of κ nontrivially because $\alpha \in \mathcal{A}$. Consider the **first** such early occurrence of κ in $\sigma = \alpha\,\kappa$. Then there is a nonempty suffix $\gamma \in \mathcal{C}$ of κ, and a nonempty prefix η of κ, and a (possibly empty) prefix ρ of α such that

$$\sigma = \alpha\,\kappa = \rho\,\eta\,\kappa = \rho\,\kappa\,\gamma.$$

Since we are looking at the first early occurrence of κ in σ, the substring $\rho\,\kappa$ is in $\mathcal{B}$. Moreover, this argument shows that these substrings ρ, η, and γ are determined uniquely from σ. This shows that every $\sigma \in \mathcal{A}\,\kappa$ is either in $\mathcal{B}$, or is in $\mathcal{B}\,\gamma$ for exactly one $\gamma \in \mathcal{C}$, and this decomposition is unique. That is,

$$\mathcal{A}\,\kappa = \mathcal{B} \cup \bigcup_{\gamma \in \mathcal{C}} \mathcal{B}\,\gamma$$

is a disjoint union of sets. Translating this into generating series yields

$$x^n\,A(x) \;=\; (1 + C(x))\,B(x).$$

Solving this for $B(x)$ and substituting this into the first equation gives

$$1 + 2x\,A(x) = A(x) \cdot \left(1 + \frac{x^n}{1 + C(x)}\right).$$

Solving this for $A(x)$ yields the result. □

In Section **??** we use a completely different method to solve a vast generalization of this excluded substring problem

3.4 Exercises.

Exercise 3.1. Prove Lemma 3.9.

Exercise 3.2. Let $\mathsf{A} = (10 \smile 101)$ and $\mathsf{B} = (001 \smile 100 \smile 1001)$. For each of AB and BA, is the expression unambiguous? What is the generating series (by length) of the set it produces?

Exercise 3.3. Let $\mathsf{A} = (00 \smile 101 \smile 11)$ and $\mathsf{B} = (00 \smile 001 \smile 10 \smile 110)$. Prove that A^* is unambiguous, and that B^* is ambiguous. Find the generating series (by length) for the set $\mathcal{A}^*$ produced by A^*.

Exercise 3.4. For each of the following sets of binary strings, write an unambiguous expression which produces that set.

(a) Binary strings that have no block of 0s of size 3, and no block of 1s of size 2.
(b) Binary strings that have no substring of 0s of length 3, and no substring of 1s of length 2.
(c) Binary strings in which the substring 011 does not occur.
(d) Binary strings in which the blocks of 0s have even length and the blocks of 1s have odd length.

Exercise 3.5. Let $\mathsf{G} = 0^*((11)^*1(00)^*00 \smile (11)^*11(00)^*0)^*$, and let $\mathcal{G}$ be the set of binary strings produced by G.

(a) Give a verbal description of the strings in the set $\mathcal{G}$.
(b) Find the generating series (by length) of $\mathcal{G}$.
(c) For $n \in \mathbb{N}$, let g_n be the number of strings in $\mathcal{G}$ of length n. Give a recurrence relation and enough initial conditions to uniquely determine g_n for all $n \in \mathbb{N}$.

Exercise 3.6.

(a) Show that the generating series (by length) for binary strings in which every block of 0s has length at least 2 and every block of 1s has length at least 3 is

$$\frac{(1-x+x^3)(1-x+x^2)}{1-2x+x^2-x^5}.$$

(b) Give a recurrence relation and enough initial conditions to determine the coefficients of this power series.

Exercise 3.7.

(a) For $n \in \mathbb{N}$, let h_n be the number of binary strings of length n such that each even-length block of 0s is followed by a block of exactly one 1 and each odd-length block of 0s is followed by a block of exactly two 1s. Show that

$$h_n = [x^n]\frac{1+x}{1-x^2-2x^3}.$$

(b) Give a recurrence relation and enough initial conditions to determine h_n for all $n \in \mathbb{N}$.

Exercise 3.8. Let $\mathcal{K}$ be the set of binary strings in which any block of 1s which immediately follows a block of 0s must have length at least as great as the length of that block of 0s. (Note: this is *not* a rational language.)

(a) Derive a formula for $K(x) = \sum_{\alpha \in \mathcal{K}} x^{\ell(\alpha)}$.
(b) Give a recurrence relation and enough initial conditions to determine the coefficients $[x^n]K(x)$ for all $n \in \mathbb{N}$.

Exercise 3.9.

(a) Fix an integer $m \geq 1$. Find the generating series (by length) of the set of binary strings in which no block has length greater than m.
(b) Fix integers $m, k \geq 1$. Find the generating series (by length) of the set of binary strings in which no block of 0s has length greater than m, and no block of 1s has length greater than k.

Exercise 3.10. Let $\mathcal{L}$ be the set of binary strings in which each block of 1s has odd length, and which do not contain the substring 0001. Let $\mathcal{L}_n$ be the set of strings in $\mathcal{L}$ of length n, and let $L(x) = \sum_{n=0}^{\infty} |\mathcal{L}_n| x^n$.

(a) Give an expression that produces the set $\mathcal{L}$ unambiguously, and explain briefly why it is unambiguous and produces $\mathcal{L}$.

(b) Use your expression from part (a) to show that

$$L(x) = \frac{1 + x - x^2}{1 - x - 2x^2 + x^3 + x^4}.$$

Exercise 3.11. Let $\mathcal{M}$ be the set of binary strings in which each block of 0s has length at most two, and which do not contain 00111 as a substring. Let $\mathcal{M}_n$ be the set of strings in $\mathcal{M}$ of length n, and let $M(x) = \sum_{n=0}^{\infty} |\mathcal{M}_n| x^n$.

(a) Give an expression that produces the set $\mathcal{M}$ unambiguously, and explain briefly why it is unambiguous and produces $\mathcal{M}$.
(b) Use your expression from part (a) to show that

$$M(x) = \frac{1 + x + x^2}{1 - x - x^2 - x^3 + x^5}.$$

Exercise 3.12. Let $\mathcal{N}$ be the set of binary strings in which each block of 0s has odd length, and each block of 1s has length one or two. Let $\mathcal{N}_n$ be the set of strings in $\mathcal{N}$ of length n, and let $N(x) = \sum_{n=0}^{\infty} |\mathcal{N}_n| x^n$.

(a) Show that

$$N(x) = \frac{1 + 2x + x^2 - x^4}{1 - 2x^2 - x^3} = -2 + x + \frac{3 + x - 3x^2}{1 - 2x^2 - x^3}.$$

(b) Derive an exact formula for $|\mathcal{N}_n|$ as a function of n.

Exercise 3.13. For $n \in \mathbb{N}$, let p_n be the number of binary strings of length n in which every block of 0s is followed by a block of 1s with the same parity of length.

(a) Determine the generating series $P(x) = \sum_{n=0}^{\infty} p_n x^n$.
(b) Show that if $n \geq 2$, then $p_n = 2 \cdot 3^{\lfloor n/2 \rfloor - 1}$.

Exercise 3.14.

(a) Let $\mathcal{Q}$ be the set of binary strings that do not contain 11000 as a substring. For $n \in \mathbb{N}$, let $\mathcal{Q}_n$ be the set of strings in $\mathcal{Q}$ of length n. Obtain a formula for the generating series $Q(x) = \sum_{n=0}^{\infty} |\mathcal{Q}_n| x^n$, with a brief explanation.

(b) Let $\mathcal{R}$ be the set of compositions, of any length, in which each part is at most 4. For $n \in \mathbb{N}$, let $\mathcal{R}_n$ be the set of compositions in $\mathcal{R}$ of size n. Obtain a formula for the generating series $R(x) = \sum_{n=0}^{\infty} |\mathcal{R}_n| x^n$, with a brief explanation.

(c) Deduce that for all integers $n \geq 1$, $|\mathcal{R}_n| = |\mathcal{Q}_n| - |\mathcal{Q}_{n-1}|$.

(d)* Part (c) implies that for every integer $n \geq 1$, there is a bijection $\mathcal{Q}_n \rightleftharpoons \mathcal{R}_n \cup \mathcal{Q}_{n-1}$. Can you determine such a bijection precisely?

Exercise 3.15. Let $\mathcal{V}$ be the set of binary strings that do not contain 0110 as a substring. Show that the generating series (by length) for $\mathcal{V}$ is

$$V(x) = \Phi_{\mathcal{V}}(x) = \frac{1 + x^3}{1 - 2x + x^3 - x^4}.$$

Exercise 3.16.

(a) Let $\mathcal{W}$ be the set of binary strings that do not contain 0101 as a substring. Obtain a formula for the generating series (by length) of $\mathcal{W}$.

(b) Fix a positive integer $r \geq 1$ and consider the binary string $(01)^r$. (Part (a) is the case $r = 2$.) Obtain a formula for the generating series of the set of binary strings that do not contain $(01)^r$.

Exercise 3.17. Let $\mathsf{S} = \mathsf{A}^*\mathsf{B}$ be an unambiguous prefix decomposition producing some set of strings $\mathcal{S} \subseteq \{0,1\}^*$. Show that the recursion $\mathsf{R} = \mathsf{B} \smile \mathsf{A}\,\mathsf{R}$ defines an expression R that produces the same set of strings $\mathcal{S} \subseteq \{0,1\}^*$. Also check that both S and R lead to the rational function $B(x)/(1 - A(x))$.

Chapter 4

Recurrence Relations.

In Chapters 2 and 3 we saw how to encode a sequence of numbers as the coefficients of a power series $G(x) = \sum_{n=0}^{\infty} g_n x^n$. We used the Sum, Product, and String Lemmas to obtain algebraic formulas for these generating series. In this section we will see two techniques for using these algebraic formulas to compute the coefficients g_n, which are the numbers we really want. First we will do the example of Fibonacci numbers in detail, and then we will develop the theory in general.

4.1 Fibonacci Numbers.

Example 4.1. The sequence of Fibonacci numbers $\mathbf{f} = (f_0, f_1, f_2, f_3, ...)$ is defined by putting $f_0 = 1$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. One can use this information to compute f_n iteratively for as long as you want:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
f_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

Can we obtain a formula for f_n as a function of n?

The answer is "yes", of course – we obtain such a formula in this section. Moreover, this is just the first example of a very general technique, which is the main subject of this chapter.

Example 4.2. We start by finding a formula for the generating series $F(x) = \sum_{n=0}^{\infty} f_n x^n$. From the information defining the Fibonacci numbers, we see that

$$F(x) = f_0 + f_1 x + \sum_{n=2}^{\infty} f_n x^n = 1 + x + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) x^n.$$

The next step is to write the RHS in terms of $F(x)$.

$$\begin{aligned} F(x) &= 1 + x + \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n \\ &= 1 + x + x \sum_{i=1}^{\infty} f_i x^i + x^2 \sum_{j=0}^{\infty} f_j x^j \\ &= 1 + x + x(F(x) - f_0) + x^2 F(x) \\ &= 1 + xF(x) + x^2 F(x). \end{aligned}$$

This equation can be solved for $F(x)$, yielding

$$F(x) = \frac{1}{1 - x - x^2}.$$

We have seen this generating series before, relating to the sets of compositions $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{J}$ in Example 2.21. Obtaining a formula for Fibonacci numbers will thus solve the counting problem for each of these sets of compositions.

The key is the denominator of the series, in this case $1-x-x^2$.

Example 4.3. We factor the denominator in the form

$$1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$$

for some complex numbers α and β, called the *inverse roots* of the polynomial. To do this we can use the Quadratic Formula, but since we are looking for the **inverse roots** of a polynomial we have to be careful. Substitute $x = 1/t$ and multiply both sides by t^2 to get

$$t^2 - t - 1 = (t - \alpha)(t - \beta).$$

Now the inverse roots of the denominator $1 - x - x^2$ are the (ordinary) roots of this *auxiliary polynomial* $t^2 - t - 1$. By the Quadratic Formula, they are

$$\left.\begin{array}{c}\alpha\\ \beta\end{array}\right\} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{5}}{2}.$$

Example 4.4. Having found the inverse roots of the denominator, the next step is to apply the Partial Fractions Theorem 4.12, which will be explained (and proved) in Section 4.3. In this case it implies that there are complex numbers A and B such that

$$\frac{1}{1 - x - x^2} = \frac{1}{(1 - \alpha x)(1 - \beta x)} = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x}.$$

There are a few different ways to find the numbers A and B, as we will see. Here we can multiply by $1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$ and collect like powers of x:

$$1 = A(1 - \beta x) + B(1 - \alpha x) = (A + B) - (A\beta + B\alpha)x.$$

Comparing coefficients, we see that $A + B = 1$ and $A\beta + B\alpha = 0$.
From $A\alpha + B\alpha = \alpha$ and $A\beta + B\alpha = 0$, we see that

$$A = \frac{\alpha}{\alpha - \beta} = \frac{5 + \sqrt{5}}{10}.$$

Similarly, from $A\beta + B\beta = \beta$ and $A\beta + B\alpha = 0$, we see that

$$B = \frac{\beta}{\beta - \alpha} = \frac{5 - \sqrt{5}}{10}.$$

Now all that remains is to put the pieces of this calculation together.

Example 4.5. We apply the geometric series expansion to the result of the partial fractions decomposition. (More generally, we would use binomial series.)

$$\frac{A}{1-\alpha x}+\frac{B}{1-\beta x}=A\sum_{n=0}^{\infty}\alpha^n x^n+B\sum_{n=0}^{\infty}\beta^n x^n=\sum_{n=0}^{\infty}\left(A\alpha^n+B\beta^n\right)x^n.$$

It follows that for all $n\in\mathbb{N}$, the Fibonacci numbers are given by the formula

$$\begin{aligned} f_n=[x^n]F(x) &= [x^n]\frac{1}{1-x-x^2} = A\alpha^n+B\beta^n \\ &= \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n+\frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n. \end{aligned}$$

That seems kind of weird, since we know that the Fibonacci numbers are integers. But notice that $\beta=(1-\sqrt{5})/2\approx -0.618$ so that as $n\to\infty$, $\beta^n\to 0$. In fact, for all $n\in\mathbb{N}$, f_n is the integer closest to $\frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n$.

4.2 Homogeneous Linear Recurrence Relations.

Definition 4.6 (Homogeneous linear recurrence relation)**.** Let $\mathrm{g}=(g_0,g_1,g_2,\ldots)$ be an infinite sequence of complex numbers. Let a_1, a_2, ..., a_d be in $\mathbb{C}$, and let $N\geq d$ be an integer. We say that g satisfies a *homogeneous linear recurrence relation* provided that

$$g_n\ +\ a_1g_{n-1}\ +\ a_2g_{n-2}\ +\cdots+\ a_dg_{n-d}\ =\ 0$$

for all $n\geq N$. The values g_0, g_1, ..., g_{N-1} are the *initial conditions* of the recurrence. The relation is *linear* because the LHS is a linear combination of the entries of the sequence g; it is *homogeneous* because the RHS of the equation is zero.

In Definition 4.6, if the RHS of the equation is instead a non-zero function $p:\mathbb{N}\to\mathbb{C}$, then this is an *inhomogeneous* linear recurrence relation.

The recurrence relation can be rewritten as

$$g_n=-(a_1g_{n-1}+a_2g_{n-2}+\cdots+a_dg_{n-d})$$

for all $n\geq N$, and this can be used to compute the numbers g_n by induction on n, using the initial conditions as the basis of induction.

Consider the Fibonacci numbers from Example 4.1: $f_0 = f_1 = 1$ are the initial conditions, and $f_n - f_{n-1} - f_{n-2} = 0$ for all $n \geq 2$ is a homogeneous linear recurrence relation. We derived the formula

$$F(x) = \sum_{n=0}^{\infty} f_n x^n = \frac{1}{1 - x - x^2}$$

for the generating series. This is an instance of a general fact about a sequence with a homogeneous linear recurrence relation. Here is another example before we see the general theory.

Example 4.7. Define a sequence $\mathbf{g} = (g_0, g_1, ...)$ by the initial conditions $g_0 = 2$, $g_1 = 5$, and $g_2 = 6$, and the relation $g_n - 3g_{n-2} - 2g_{n-3} = 0$ for all $n \geq 3$. Obtain a formula for the generating series $G(x) = \sum_{n=0}^{\infty} g_n x^n$.

The general method is to multiply the recurrence by x^n and sum over all $n \geq N$. In this case

$$\sum_{n=3}^{\infty} (g_n - 3g_{n-2} - 2g_{n-3})\, x^n = 0.$$

Now we split the LHS into separate summations, reindex them, and write everything in terms of the power series $G(x)$

$$\begin{aligned}
\sum_{n=3}^{\infty} g_n x^n - 3 \sum_{n=3}^{\infty} g_{n-2} x^n - 2 \sum_{n=3}^{\infty} g_{n-3} x^n &= 0 \\
(G(x) - g_0 - g_1 x - g_2 x^2) - 3x^2 \sum_{j=1}^{\infty} g_j x^j - 2x^3 \sum_{k=0}^{\infty} g_k x^k &= 0 \\
(G(x) - 2 - 5x - 6x^2) \;-\; 3x^2\,(G(x) - 2) \;-\; 2x^3\, G(x) &= 0 \\
G(x) \;-\; 3x^2\, G(x) \;-\; 2x^3\, G(x) &= 2 + 5x.
\end{aligned}$$

It follows that

$$G(x) = \frac{2 + 5x}{1 - 3x^2 - 2x^3}.$$

Notice how the polynomial $1 - 3x^2 - 2x^3$ in the denominator of this formula is related to the linear recurrence relation $g_n - 3g_{n-2} - 2g_{n-3} = 0$ for $n \geq 3$. We can explain the numerator, too, if we make the convention that

$g_n = 0$ for all integers $n < 0$. Then, using the initial conditions, we have

$$\begin{aligned} g_0 - 3g_{-2} - 2g_{-3} &= 2 - 0 - 0 = 2 \text{ for } n = 0, \\ g_1 - 3g_{-1} - 2g_{-2} &= 5 - 0 - 0 = 5 \text{ for } n = 1, \\ g_2 - 3g_0 - 2g_{-1} &= 6 - 3 \cdot 2 - 0 = 0 \text{ for } n = 2, \\ g_n - 3g_{n-2} - 2g_{n-3} &= 0 \text{ for } n \geq 3. \end{aligned}$$

Theorem 4.8. *Let $\mathbf{g} = (g_0, g_1, g_2, \ldots)$ be a sequence of complex numbers, and let $G(x) = \sum_{n=0}^{\infty} g_n x^n$ be the corresponding generating series. The following are equivalent.*

(a) The sequence $\mathbf{g}$ satisfies a homogeneous linear recurrence relation

$$g_n + a_1 g_{n-1} + \cdots + a_d g_{n-d} = 0 \text{ for all } n \geq N,$$

with initial conditions g_0, g_1, ..., g_{N-1}.

(b) The series $G(x) = P(x)/Q(x)$ is a quotient of two polynomials. The denominator is

$$Q(x) = 1 + a_1 x + a_2 x^2 + \cdots + a_d x^d$$

and the numerator is $P(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1} x^{N-1}$, in which

$$b_k = g_k + a_1 g_{k-1} + \cdots + a_d g_{k-d}$$

for all $0 \leq k \leq N-1$, with the convention that $g_n = 0$ for all $n < 0$.

Proof. To prove this theorem, we just copy the calculation in Example 4.7, but do it in the most general case. For convenience, let $a_0 = 1$. Assume that part (a) holds, and let

$$Q(x) = 1 + a_1 x + a_2 x^2 + \cdots + a_d x^d.$$

Consider the product $Q(x)G(x)$:

$$\begin{aligned} Q(x)G(x) &= \left(\sum_{j=0}^{d} a_j x^j \right) \left(\sum_{n=0}^{\infty} g_n x^n \right) \\ &= \sum_{j=0}^{d} \sum_{n=0}^{\infty} a_j g_n x^{n+j} = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{d} a_j g_{k-j} \right) x^k. \end{aligned}$$

In the last step we have re-indexed the double sum using $k = n + j$, and used the convention that $g_n = 0$ for all $n < 0$.

The coefficient of x^k in this formula is $g_k + a_1 g_{k-1} + \cdots + a_d g_{k-d}$. This is the LHS of the recurrence relation for g applied when $n = k$. Thus, this coefficient is zero for $k \geq N$. On the other hand, for $0 \leq k \leq N-1$, we see that it is $\sum_{j=0}^{d} a_j g_{k-j} = b_k$ by the way the numbers b_k are defined. That is,

$$Q(x)G(x) = \sum_{k=0}^{N-1} b_k x^k = P(x),$$

and it follows that $G(x) = P(x)/Q(x)$. This shows that (a) implies (b).

Conversely, assume that (b) holds and that $G(x) = P(x)/Q(x)$ is as given. We essentially run the argument for the first part of the proof in reverse. The equations $b_k = g_k + a_1 g_{k-1} + \cdots + a_d g_{k-d}$ for $0 \leq k \leq N-1$ (with the convention that $g_n = 0$ for $n < 0$) determine the initial conditions g_0, g_1,... g_{N-1} inductively. For $n \geq N$, the coefficient of x^n in $P(x) = Q(x)G(x)$ is zero. This implies that $g_k + a_1 g_{k-1} + \cdots + a_d g_{k-d} = 0$ for all $k \geq N$, showing that (b) implies (a). □

Theorem 4.8 is useful in both directions, as the following two examples show.

Example 4.9. Let $D(x) = \sum_{n=0}^{\infty} d_n x^n = \dfrac{1 - 3x + 4x^2}{1 - 2x + 3x^3}$. Obtain a homogeneous linear recurrence relation and initial conditions satisfied by the sequence $\mathbf{d} = (d_0, d_1, d_2, ...)$.

From Theorem 4.8 we can read from the RHS that for all $n \in \mathbb{N}$:

$$d_n - 2d_{n-1} + 3d_{n-3} = \begin{cases} 1 & \text{if } n = 0, \\ -3 & \text{if } n = 1, \\ 4 & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases}$$

with the convention that $d_n = 0$ if $n < 0$. We can determine the initial conditions inductively as follows: $d_0 = 1$; $d_1 - 2d_0 = -3$, so $d_1 = -1$; $d_2 - 2d_1 = 4$, so $d_2 = 2$. The recurrence $d_n - 2d_{n-1} + 3d_{n-3} = 0$ holds for all $n \geq 3$. Using the initial conditions $d_0 = 1$, $d_1 = -1$, $d_2 = 2$, and the recurrence $d_n = 2d_{n-1} - 3d_{n-3}$ for all $n \geq 3$, we can compute d_n for as long as we want:

n	0	1	2	3	4	5	6	7	8
d_n	1	-1	1	1	5	4	5	-5	-22

Example 4.10. A sequence $\mathsf{s} = (s_0, s_1, s_2, ...)$ is defined by the initial conditions $s_0 = 1$, $s_1 = 2$, $s_2 = 1$, and the recurrence $s_n - s_{n-1} - 2s_{n-3} = 0$ for all $n \geq 3$. Obtain a formula for the generating series $S(x) = \sum_{n=0}^{\infty} s_n x^n$.

Since we have the information at hand, we might as well compute a few more values of s_n:

n	0	1	2	3	4	5	6	7
s_n	1	2	1	3	7	9	15	29

To get the generating series, Theorem 4.8 implies immediately that the denominator is $1 - x - 2x^3$. To obtain the numerator we apply the recurrence for small values of n, with the convention that $s_n = 0$ if $n < 0$.

$$s_n - s_{n-1} - 2s_{n-3} = \begin{cases} 1 & \text{if } n = 0, \\ 2 - 1 = 1 & \text{if } n = 1, \\ 1 - 2 = -1 & \text{if } n = 2. \end{cases}$$

Thus, the numerator is $1 + x - x^2$. The generating series is

$$S(x) = \frac{1 + x - x^2}{1 - x - 2x^3}.$$

Example 4.11. Let's revisit Example 2.22, concerning the set $\mathcal{Q}$ of all compositions in which each part is at least two, and the number of parts is even. We derived the generating series

$$Q(x) = \sum_{n=0}^{\infty} q_n x^n = \frac{1 - 2x + x^2}{1 - 2x + x^2 - x^4}.$$

From Theorem 4.8 we see immediately that

$$q_n - 2q_{n-1} + q_{n-2} - q_{n-4} = \begin{cases} 1 & \text{if } n = 0, \\ -2 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases}$$

with the convention that $q_n = 0$ if $n < 0$. We can inductively calculate the first several values of q_n:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
q_n	1	0	0	0	1	2	3	4	6	10	17	28	45	72	116

We have determined that $|\mathcal{Q}_{14}| = 116$, but we have not listed all these compositions individually. That is pretty cool, when you think about it.

4.3 Partial Fractions.

A *rational function* is a quotient of two polynomials $P(x)/Q(x)$. This is analogous to a rational number being a quotient of two integers.

Theorem 4.12 (Partial Fractions)**.** *Let $G(x) = P(x)/Q(x)$ be a rational function in which $\deg P < \deg Q$ and the constant term of $Q(x)$ is 1. Factor the denominator to obtain its inverse roots:*

$$Q(x) = (1 - \lambda_1 x)^{d_1}(1 - \lambda_2 x)^{d_2} \cdots (1 - \lambda_s x)^{d_s}$$

in which λ_1, ..., λ_s are distinct nonzero complex numbers and $d_1 + \cdots + d_s = d = \deg Q$. Then there are d complex numbers:

$$C_1^{(1)}, C_1^{(2)}, ..., C_1^{(d_1)};\ \ C_2^{(1)}, C_2^{(2)}, ..., C_2^{(d_2)};\ \ ...\ ;\ \ C_s^{(1)}, C_s^{(2)}, ..., C_s^{(d_s)}$$

(which are uniquely determined) such that

$$G(x) = \frac{P(x)}{Q(x)} = \sum_{i=1}^{s} \sum_{j=1}^{d_s} \frac{C_i^{(j)}}{(1 - \lambda_i x)^j}.$$

Proof. Consider the set $\mathcal{V}_Q$ of all rational functions $P(x)/Q(x)$ in which Q is a fixed polynomial as in the statement of the theorem, and P is any polynomial of degree strictly less than $d = \deg Q$. It is easily seen that $\mathcal{V}_Q$ is a vector space over the complex numbers $\mathbb{C}$, since if $P(x)$ and $R(x)$ both have degree less than d and $\alpha \in \mathbb{C}$ then $P(x) + \alpha R(x)$ has degree less than d. It is also clear that the vectors

$$\frac{1}{Q(x)}, \quad \frac{x}{Q(x)}, \quad \frac{x^2}{Q(x)}, \quad \ldots, \quad \frac{x^{d-1}}{Q(x)}$$

span $\mathcal{V}_Q$ as a vector space over $\mathbb{C}$. Thus, the dimension of $\mathcal{V}_Q$ is at most d.

Now we claim that for every $1 \leq i \leq s$ and $1 \leq j \leq d_i$, the quotient $1/(1-\lambda_i x)^j$ is in $\mathcal{V}_Q$. This is because

$$\frac{1}{(1-\lambda_i x)^j} = \frac{(1-\lambda_i x)^{d_i - j} \prod_{h \neq i} (1-\lambda_h x)^{d_h}}{Q(x)}$$

and the numerator has degree $d - j \leq d - 1 < d$.

The essential point in the proof is that the set of vectors

$$\mathcal{B} \;=\; \left\{ \frac{1}{(1-\lambda_i x)^j} \;:\; 1 \leq i \leq s \;\text{ and }\; 1 \leq j \leq d_i \right\}$$

in $\mathcal{V}_Q$ is linearly independent. From this we can conclude that the dimension of $\mathcal{V}_Q$ is at least $d_1 + \cdots + d_s = d$. It then follows that $\dim \mathcal{V}_Q = d$ and that $\mathcal{B}$ is a basis for $\mathcal{V}_Q$. Therefore, every vector in $\mathcal{V}_Q$ can be written uniquely as a linear combination of vectors in $\mathcal{B}$. That is exactly what the Partial Fractions Theorem is claiming.

It remains only to show that $\mathcal{B}$ is a linearly independent set. Consider any linear combination of vectors in $\mathcal{B}$ which gives the zero vector:

$$\sum_{i=1}^{s} \sum_{j=1}^{d_s} \frac{C_i^{(j)}}{(1-\lambda_i x)^j} \;=\; 0. \tag{4.1}$$

We must show that $C_i^{(j)} = 0$ for all $1 \leq i \leq s$ and $1 \leq j \leq d_i$. Suppose not. Then there is some index $1 \leq p \leq s$ for which at least one of the coefficients $C_p^{(1)}$, $C_p^{(2)}$, ..., $C_p^{(d_p)}$ is not zero. Letting $C_p^{(t)} \neq 0$ be the one with the largest superscript, we also have $C_p^{(t+1)} = \cdots = C_p^{(d_p)} = 0$.

Now multiply equation (4.1) by $(1-\lambda_p x)^t$. Separating out the terms with $i = p$ and using the fact that $C_p^{(t+1)} = \cdots = C_p^{(d_p)} = 0$, we see that

$$\sum_{j=1}^{t} C_p^{(j)} (1-\lambda_p x)^{t-j} \; + \; \sum_{i \neq p} \sum_{j=1}^{d_i} C_i^{(j)} \, \frac{(1-\lambda_p x)^t}{(1-\lambda_i x)^j} \; = \; 0.$$

The LHS is a rational function of the variable x which does not have a pole at the point $x = 1/\lambda_p$, so we can substitute this value for x. But every term on the LHS has a factor of $(1-\lambda_p x)$ except for the term with $i = p$ and $j = t$. Thus, upon making the substitution $x = 1/\lambda_p$ this equation becomes

$$C_p^{(t)} \; = \; 0.$$

But this contradicts our choice of p and t. This contradiction shows that all the coefficients $C_i^{(j)}$ in equation (4.1) must be zero, and it follows that the set $\mathcal{B}$ is linearly independent.

Since $\mathcal{B}$ is a set of d linearly independent vectors in a vector space $\mathcal{V}_Q$ of dimension at most d, it follows that $\mathcal{B}$ is a basis for $\mathcal{V}_Q$, and the proof is complete. $\square$

Example 4.13. Let's re-examine the generating series

$$G(x) = \frac{2+5x}{1-3x^2-2x^3}$$

from Example 4.7. This satisfies the hypotheses of the Partial Fractions Theorem 4.12. The denominator $1-3x^2-2x^3$ vanishes when $x = -1$, so that $1 + x$ is a factor. Some calculation shows that

$$1-3x^2-2x^3 = (1+x)(1-x-2x^2) = (1+x)^2(1-2x).$$

Thus, there are complex numbers A, B, C such that

$$\frac{2+5x}{1-3x^2-2x^3} = \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{1-2x}.$$

Now multiply by the denominator on the LHS.

$$2+5x = A(1+x)(1-2x) + B(1-2x) + C(1+x)^2.$$

This is an equality of polynomials, so it holds for any value of x.

- At $x = -1$ we find that $2 - 5 = B(1 + 2)$, so that $B = -1$.
- At $x = 1/2$ we find that $2 + 5/2 = C(3/2)^2$, so that $9/2 = C(9/4)$, so that $C = 2$.
- At $x = 0$ we find that $2 = A + B + C$, so that $A = 2 - B - C = 2 + 1 - 2 = 1$.

Therefore
$$\frac{2 + 5x}{1 - 3x^2 - 2x^3} = \frac{1}{1 + x} - \frac{1}{(1 + x)^2} + \frac{2}{1 - 2x}.$$

Now we can expand each of these terms using Binomial Series, and collect the results.

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty}(-x)^n - \sum_{n=0}^{\infty}\binom{n+1}{1}(-x)^n + 2\sum_{n=0}^{\infty}(2x)^n \\ &= \sum_{n=0}^{\infty}\left((-1)^n - (n+1)(-1)^n + 2\cdot 2^n\right)x^n \\ &= \sum_{n=0}^{\infty}\left(2^{n+1} - n(-1)^n\right)x^n. \end{aligned}$$

It follows that $g_n = [x^n]G(x) = 2^{n+1} + n(-1)^{n+1}$ for all $n \in \mathbb{N}$. This can be "reality checked" by comparison with the initial conditions $g_0 = 2$, $g_1 = 5$, and $g_2 = 6$, and the recurrence relation $g_n - 3g_{n-2} - 2g_{n-3} = 0$ for all $n \geq 3$ defining this sequence in Example 4.7. The first few values are

n	0	1	2	3	4	5	6
g_n	2	5	6	19	28	69	122

4.3.1 The Main Theorem.

Theorem 4.14. *Let* $\mathbf{g} = (g_0, g_1, g_2)$ *be a sequence of complex numbers, and let* $G(x) = \sum_{n=0}^{\infty} g_n x^n$ *be the corresponding generating series. Assume that the equivalent conditions of Theorem 4.8 hold, and that*

$$G(x) = R(x) + \frac{P(x)}{Q(x)}$$

for some polynomials $P(x)$, $Q(x)$, *and* $R(x)$ *with* $\deg P(x) < \deg Q(x)$ *and* $Q(0) = 1$. *Factor* $Q(x)$ *to obtain its inverse roots and their multiplicities:*

$$Q(x) = (1 - \lambda_1 x)^{d_1}(1 - \lambda_2 x)^{d_2} \cdots (1 - \lambda_s x)^{d_s}.$$

Then there are polynomials $p_i(n)$ *for* $1 \leq i \leq s$, *with* $\deg p_i(n) < d_i$, *such that for all* $n > \deg R(x)$,

$$g_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n + \cdots + p_s(n)\lambda_s^n.$$

Proof. The conclusion of the theorem only concerns terms with $n > \deg R(x)$, so we can basically ignore the polynomial $R(x)$. In truth, all it is doing is getting in the way, and preventing the formula from holding for smaller values of n. So we are going to concentrate on the quotient $P(x)/Q(x)$, to which the Partial Fractions Theorem 4.12 applies.

Consider the factor $(1 - \lambda_i x)^{d_i}$ of the denominator $Q(x)$. In the partial fractions expansion of $P(x)/Q(x)$, this contributes

$$\frac{C_i^{(1)}}{1 - \lambda_i x} + \frac{C_i^{(2)}}{(1 - \lambda_i x)^2} + \cdots + \frac{C_i^{(d_i)}}{(1 - \lambda_i x)^{d_i}}.$$

Each term is a binomial series, and can be expanded accordingly:

$$\begin{aligned}\sum_{j=1}^{d_i} \frac{C_i^{(j)}}{(1 - \lambda_i x)^j} &= \sum_{j=1}^{d_i} C_i^{(j)} \sum_{n=0}^{\infty} \binom{n+j-1}{j-1} \lambda_i^n x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=1}^{d_i} C_i^{(j)} \binom{n+j-1}{j-1} \right) \lambda_i^n x^n.\end{aligned}$$

Notice that $\binom{n+j-1}{j-1}$ is a polynomial function of n of degree $j - 1$. It follows that

$$p_i(n) \;=\; \sum_{j=1}^{d_i} C_i^{(j)} \binom{n+j-1}{j-1}$$

is a polynomial function of n of degree at most $d_i - 1$. The contribution of the inverse root λ_i to the coefficient $g_n = [x^n]G(x)$ is thus $p_i(n)\lambda_i^n$. By the form of the partial fractions expansion we see that

$$g_n = p_1(n)\lambda_1^n \;+\; p_2(n)\lambda_2^n \;+\; \cdots \;+\; p_s(n)\lambda_s^n,$$

completing the proof. □

The converse of Theorem 4.14 also holds – see Exercise 4.11.

One can use Theorem 4.14 to go straight from a recurrence relation to a formula for its entries, without doing Partial Fractions explicitly. Here is an example of this kind of calculation.

Example 4.15. *A sequence* $\mathbf{h}$ *of integers is given by the initial conditions* $h_0 = 1$, $h_1 = 1$, $h_2 = 0$, $h_3 = 2$, $h_4 = -4$, $h_5 = 3$, *and the recurrence* $h_n - 3h_{n-1} + 4h_{n-3} = 0$ *for all* $n \geq 6$. *Obtain a formula for* h_n *as a function of* n.

From Theorem 4.8 we see that the denominator of the generating series $H(x) = h_n x^n$ is $1 - 3x + 4x^3$. This vanishes at $x = -1$, so $1 + x$ is a factor. After some work, we obtain

$$1 - 3x + 4x^3 = (1 + x)(1 - 4x + 4x^2) = (1 - 2x)^2(1 + x).$$

Theorem 4.14 implies that there are constants A, B, C such that for sufficiently large n, $h_n = (A + Bn)2^n + C(-1)^n$. To determine these constants we need to take data from the sequence $\mathbf{h}$ from a point later than the degree of the polynomial $R(x)$ appearing in Theorem 4.14. From Theorem 4.8, in this case the degree of the numerator of the generating series $H(x)$ is no more than five, since the general case of the recurrence holds for $n \geq 6$. Writing

$$H(x) = R(x) + \frac{P(x)}{1 - 3x + 4x^3} = \frac{(1 - 3x + 4x^3)R(x) + P(x)}{1 - 3x + 4x^3},$$

it follows that the degree of $R(x)$ is at most two. So we can fit the form $h_n = (A + Bn)2^n + C(-1)^n$ to the data $h_3 = 2$, $h_4 = -4$, and $h_5 = 3$.

This gives three equations in three unknowns – a standard linear algebra problem.

$$\begin{aligned} h_3 = 2 = (A+3B)8 - C &= 8A + 24B - C \\ h_4 = -4 = (A+4B)16 + C &= 16A + 64B + C \\ h_5 = 3 = (A+5B)32 - C &= 32A + 160B - C \end{aligned}$$

In this case, it is a rather unpleasant linear algebra problem. Sparing you the details, the solution is $A = -5/16$, $B = 1/16$, $C = -3$, and so

$$h_n = (n-5)2^{n-4} - 3(-1)^n$$

for all $n \geq 3$. The values for h_n with $0 \leq n \leq 2$ are given in the initial conditions.

4.3.2 Inhomogeneous Linear Recurrence Relations.

Example 4.16. Define a sequence of integers $\mathbf{g} = (g_0, g_1, g_2, \ldots)$ by the initial conditions $g_0 = 1$ and $g_1 = 2$, and the recurrence relation

$$g_n = g_{n-1} + 2\,g_{n-2} - n + 1$$

for all $n \geq 2$. This clearly determines the sequence g inductively:

n	0	1	2	3	4	5	6	7	8
g_n	1	2	3	5	8	14	25	47	90

What is g_n as a function of $n \in \mathbb{N}$?

We solve Example 4.16 by generalizing the method above just a little bit. First, write the recurrence in the form

$$g_n - g_{n-1} - 2\,g_{n-2} = -n + 1$$

for all $n \geq 2$. This is an **inhomogeneous** linear recurrence relation according to Definition 4.6. But we can proceed just as before. We begin by obtaining a formula for the generating series $G(x) = \sum_{n=0}^{\infty} g_n x^n$. Multiply both sides by x^n and sum over all $n \geq 2$. On the RHS we obtain

$$\sum_{n=2}^{\infty} (-n+1)\, x^n = \sum_{j=0}^{\infty} (-(j+2)+1)\, x^{j+2} \;=\; -x^2 \sum_{j=0}^{\infty} (j+1)\, x^j$$
$$= \frac{-x^2}{(1-x)^2}.$$

On the LHS we obtain

$$\begin{aligned}
&\sum_{n=2}^{\infty} (g_n \;-\; g_{n-1} \;-\; 2\, g_{n-2})\, x^n \\
&= \sum_{n=2}^{\infty} g_n\, x^n \;-\; \sum_{n=2}^{\infty} g_{n-1}\, x^n -\; 2 \sum_{n=2}^{\infty} g_{n-2}\, x^n \\
&= (G(x) - g_0 - g_1 x) \;-\; x \sum_{j=1}^{\infty} g_j\, x^j \;-\; 2\, x^2 \sum_{k=0}^{\infty} g_k\, x^k \\
&= (G(x) - 1 - 2x) - x\,(G(x) - 1) - 2x^2\, G(x) \\
&= (1 - x - 2x^2) G(x) - 1 - x.
\end{aligned}$$

Equating the LHS and the RHS yields

$$(1 - x - 2x^2) G(x) \;=\; 1 + x - \frac{x^2}{(1-x)^2} = \frac{(1+x)(1-x)^2 - x^2}{(1-x)^2}$$

Noting that $1 - x - 2x^2 = (1+x)(1-2x)$, we obtain the formula

$$G(x) \;=\; \frac{1 - x - 2x^2 + x^3}{(1+x)(1-x)^2(1-2x)}.$$

This is a rational function, and so the Main Theorem 4.14 applies. There are complex numbers A, B, C, D such that $g_n = A(-1)^n + (B + Cn) + D2^n$ for all $n \in \mathbb{N}$. For $n \in \{0, 1, 2, 3\}$ this yields the system of linear equations

$$\begin{array}{rrrrl}
A & +B & & +D & = 1 \\
-A & +B & +C & +2D & = 2 \\
A & +B & +2C & +4D & = 3 \\
-A & +B & +3C & +8D & = 5
\end{array}$$

Solving this system yields $A = -1/12$, $B = 3/4$, $C = 1/2$, $D = 1/3$, so that

$$g_n \;=\; \frac{1}{12}\, \left(2^{n+2} + (6n+9) - (-1)^n \right)$$

for all $n \in \mathbb{N}$.

Example 4.17. The denominator of $G(x)$ in the above example is $1-3x+x^2+3x^3-2x^4$. From Theorem 4.8 we see that the sequence g satisfies the **homogeneous** linear recurrence relation and initial conditions given by

$$g_n - 3g_{n-1} + g_{n-2} + 3g_{n-3} - 2g_{n-4} = \begin{cases} 1 & \text{if } n = 0, \\ -1 & \text{if } n = 1, \\ -2 & \text{if } n = 2, \\ 1 & \text{if } n = 3, \\ 0 & \text{if } n \geq 4. \end{cases}$$

This agrees with the results above.

Examples 4.16 and 4.17 illustrate a general fact: if the generating series of the RHS in an **inhomogeneous** linear recurrence relation is a rational function, then the generating series for the entries of the sequence is also a rational function. Thus, the sequence in fact satisfies a **homogeneous** linear recurrence relation, so we are actually back in the case we have already considered. Proving this in general is the main point of this subsection.

The following terminology is not standard but will be convenient. A function $q : \mathbb{N} \to \mathbb{C}$ is *polyexp* if there are polynomial functions $q_i(n)$ and complex numbers $\beta_i \in \mathbb{C}$ for $1 \leq i \leq t$ such that

$$q(n) \ = \ q_1(n)\beta_1^n \ + \ q_2(n)\beta_2^n \ + \ \cdots \ + \ q_t(n)\beta_t^n \tag{4.2}$$

for all $n \in \mathbb{N}$. For example, $\cos(n\theta) = (\mathrm{e}^{\mathrm{i}\theta n} + \mathrm{e}^{-\mathrm{i}\theta n})/2$ is polyexp, but $\sqrt{n}$ is not. More generally, the function $q : \mathbb{N} \to \mathbb{C}$ is *eventually polyexp* if there is an integer $M \in \mathbb{N}$ such that equation (4.2) holds for all $n \geq M$. The Main Theorem 4.14 thus states that if $G(x) = \sum_{n=0}^{\infty} g_n x^n$ is a rational function then g_n is an eventually polyexp function of n. (Exercise 4.11 is the converse implication.)

Theorem 4.18. *Let* $\mathbf{g} = (g_0, g_1, g_2, ...)$ *be a sequence of complex numbers. The following are equivalent.*

(a) The sequence g *satisfies a homogeneous linear recurrence relation (with initial conditions).*

(b) The sequence g *satisfies a possibly inhomogeneous linear recurrence relation (with initial conditions) in which the RHS is an eventually*

polyexp function.

(c) *The generating series $G(x) = \sum_{n=0} g_n\, x^n$ is a rational function (a quotient of polynomials in x).*

(d) *The sequence $\mathbf{g} = (g_0, g_1, g_2, ...)$ is an eventually polyexp function.*

Proof. Theorem 4.8 shows that conditions (a) and (c) are equivalent. Theorem 4.14 shows that (c) implies (d). That (d) implies (c) is left as Exercise 4.11. It is clear that (a) implies (b). All that remains is to show that (b) implies (c).

Thus, assume that g satisfies the linear recurrence relation

$$g_n \ + \ a_1\, g_{n-1} \ + \cdots + \ a_d\, g_d \ = \ q(n)$$

for all $n \geq N$, with initial conditions g_0, g_1, ..., g_{N-1}, in which $q : \mathbb{N} \to \mathbb{C}$ is an eventually polyexp function as in equation (4.2) for all $n \geq M$.

UNDER CONSTRUCTION

□

4.4 Quadratic Recurrence Relations.

Let $\mathbf{g} = (g_0, g_1, g_2, ...)$ be a sequence of numbers with generating series $G(x) = \sum_{n=0}^{\infty} g_n x^n$. In Theorem 4.8 we saw that g satisfies a homogeneous linear recurrence relation (with initial conditions) if and only if $G(x) = P(x)/Q(x)$ is a rational function. Rewriting this as $Q(x)G(x) - P(x) = 0$, we see that $G(x)$ is a solution to a linear equation: $QG - P = 0$.

Definition 4.19. The sequence g satisfies a *quadratic recurrence* if its generating series $G(x)$ satisfies a quadratic equation:

$$A(x)G(x)^2 + B(x)G(x) + C(x) = 0.$$

Here, the coefficients $A(x)$, $B(x)$, and $C(x)$ are power series in x.

There are two solutions to the equation in Definition 4.19, and they can be found using the Quadratic Formula:

$$\left.\begin{array}{l} G_+(x) \\ G_-(x) \end{array}\right\} = \frac{-B(x) \pm \sqrt{B(x)^2 - 4\, A(x)\, C(x)}}{2\, A(x)}.$$

Rigorous justification for this kind of algebra with power series is discussed in detail in CO 330. If $G(x)$ is a generating series for some combinatorial objects then it has only nonnegative coefficients and nonnegative exponents. This can be used to decide which case of the $\pm$ sign to take. In general, only one of $G_+(x)$ or $G_-(x)$ is the correct generating series.

4.4.1 The general binomial series.

In Section 2.1 we saw the Binomial Theorem and The Binomial Series with negative integer exponents. That is, for a natural number $n \in \mathbb{N}$,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

and for a positive integer $t \geq 1$,

$$\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n.$$

These are two special cases of the general *binomial series*.

Definition 4.20. For any complex number $\alpha \in \mathbb{C}$ and nonnegative integer $k \in \mathbb{N}$, the *k-th binomial coefficient of α* is

$$\binom{\alpha}{k} = \frac{1}{k!}(\alpha)(\alpha-1)\cdots(\alpha-k+1).$$

This binomial coefficient is a polynomial function of α of degree k.

Theorem 4.21 (The Binomial Series)**.** *For any complex number $\alpha \in \mathbb{C}$,*

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

Sketch of proof. We can think of $(1+x)^\alpha$ as a function of a complex variable x. The only possible singularity is that 0^α might not be well-defined – this can happen only for $x = -1$. Therefore, $(1+x)^\alpha$ is analytic in the disc $|x| < 1$, and so it has a Taylor series expansion. By Taylor's Theorem, the coefficient of x^k in this Taylor series expansion is

$$\frac{1}{k!}\,\frac{\mathrm{d}^k}{\mathrm{d}x^k}(1+x)^\alpha\bigg|_{x=0} = \frac{1}{k!}(\alpha)(\alpha-1)\cdots(\alpha-k+1)(1+x)^{\alpha-k}\bigg|_{x=0} = \binom{\alpha}{k}.$$

This proves the stated formula. □

We will use the following special case.

Proposition 4.22. $\sqrt{1-4x} \;=\; 1-2\sum_{k=1}^{\infty}\frac{1}{k}\binom{2k-2}{k-1}x^k.$

Proof. By Theorem 4.21, $\sqrt{1-4x} \;=\; \sum_{k=0}^{\infty}(-1)^k 4^k\binom{1/2}{k}x^k.$

For $k=0$, the coefficient of x^0 is $(-1)^0 4^0\binom{1/2}{0}=1$.

For $k\geq 1$, we can calculate as follows.

$$\begin{aligned}(-1)^k4^k\binom{1/2}{k} &= (-1)^k4^k\,\frac{1}{k!}\,(1/2)(-1/2)(-3/2)\cdots(1/2-k+1)\\ &= -4^k\,\frac{1}{k!}\,(1/2)(1/2)(3/2)\cdots(k-3/2)\\ &= -2^k\,\frac{1}{k!}\,(1)(1)(3)(5)\cdots(2k-3)\\ &= -\frac{2}{k}\cdot\frac{(1)(3)\cdots(2k-3)}{(k-1)!}\cdot\frac{(2)(4)\cdots(2k-2)}{(k-1)!}\\ &= -\frac{2}{k}\binom{2k-2}{k-1}.\end{aligned}$$

(Where did we use the fact that $k\geq 1$ in this calculation?) □

4.4.2 Catalan numbers.

The numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$ are called *Catalan numbers*. The first few Catalan numbers are shown here:

n	0	1	2	3	4	5	6	7	8	9	10
C_n	1	1	2	5	14	42	132	429	1430	4862	16796

Catalan numbers occur surprisingly often in answers to counting problems.

Example 4.23 (Well-Formed Parenthesizations.)**.**
A *well-formed parenthesization* (WFP) is a sequence of n opening parentheses and n closing parentheses which "match together" using the usual rules for grouping parentheses. The *size* of a WFP is the number of opening parentheses in it. Here are all the WFPs of size 3:

()()() **()(())** **(())()** **(()())** **((()))**

And here are all the WFPs of size 4:

()()()() **(())()()** **(()())()** **()()(())**
(())(()) **(()(()))** **()(())()**
((()))() **((())())** **()(()())** **(()()())**
((()())) **()((()))** **(((())))**

We determine the number w_n of WFPs of size n, for all $n \in \mathbb{N}$. Let

$$W(x) = \sum_{n=0}^{\infty} w_n x^n$$

be the generating series for WFPs with respect to size.

We can obtain a quadratic recurrence for $W(x)$, as follows. The empty sequence ε contributes $x^0 = 1$ to the generating series $W(x)$. Any other WFP γ begins with an opening parenthesis. There is exactly one closing parenthesis which matches to the beginning parenthesis. That is, $\gamma = \mathbf{(}\,\alpha\,\mathbf{)}\,\beta$ for some other sequences α and β. Note that α or β might be empty. Because of the way parentheses are matched to each other, both α and β are in fact WFPs themselves. The total number of opening parentheses in γ is $n(\gamma) = 1 + n(\alpha) + n(\beta)$. Conversely, given any WFPs α and β we can always form a new nonempty WFP: $\mathbf{(}\,\alpha\,\mathbf{)}\,\beta$.

Writing a 0 instead of a **(**, and a 1 instead of a **)**, a WFP can be thought of as a binary string in $\{0,1\}^*$. Let $\mathcal{W}$ be the set of all binary strings corresponding to WFPs. The previous paragraph justifies the claim that the recursive decomposition

$$\mathcal{W} = \varepsilon \smile 0\,\mathcal{W}\,1\,\mathcal{W}$$

is unambiguous. This allows us to calculate as follows:

$$\begin{aligned} W(x) = \sum_{\gamma \in \mathcal{W}} x^{n(\gamma)} &= x^{n(\varepsilon)} + \sum_{\gamma \in \mathcal{W} \smallsetminus \{\varepsilon\}} x^{n(\gamma)} \\ &= 1 + \sum_{\alpha \in \mathcal{W}} \sum_{\beta \in W} x^{1+n(\alpha)+n(\beta)} \\ &= 1 + x \left(\sum_{\alpha \in \mathcal{W}} x^{n(\alpha)} \right) \left(\sum_{\beta \in \mathcal{W}} x^{n(\beta)} \right) \\ &= 1 + xW(x)^2. \end{aligned}$$

Now we can solve this equation $xW(x)^2 - W(x) + 1 = 0$ using the Quadratic Formula:

$$\left. \begin{array}{l} W_+(x) \\ W_-(x) \end{array} \right\} = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Proposition 4.22 gives the power series for $\sqrt{1-4x}$, so that

$$\frac{1 \pm \sqrt{1-4x}}{2x} = \frac{1}{2x} \pm \frac{1}{2x} \left(1 - 2 \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^{n+1} \right).$$

To get nonnegative coefficients, and to cancel the term $1/2x$, we need to take the minus sign from the $\pm$. The result is

$$W(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n.$$

Thus, the number of WFPs of size n is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, for each $n \in \mathbb{N}$.

Since the generating series for the set $\mathcal{W}$ is $W(x) = (1 - \sqrt{1-4x})/2x$, which is not a rational function, it follows from Theorem 3.13 that the set of WFPs is not a rational language.

4.5 Exercises.

Exercise 4.1. For each of the sets of compositions from Exercise 2.15, do the following.

- Derive a recurrence relation and initial conditions for the coefficients of the corresponding generating series $G(x) = \sum_{n=0}^{\infty} g_n x^n$.
- Calculate the coefficients g_0, g_1, ... up to g_9.

Exercise 4.2. Let $\mathcal{K}$ be the set of compositions $\gamma = (c_1, c_2, ..., c_k)$ with at least one part, and such that the first part is odd. Let $K(x)$ be the generating series for $\mathcal{K}$ with respect to size.

(a) Show that
$$K(x) = \frac{x}{(1+x)(1-2x)}.$$

(b) Use part (a) to show that, among all 2^{n-1} compositions of size $n \geq 1$, the fraction of these compositions in the set $\mathcal{K}$ is
$$\frac{2}{3} + \frac{1}{3}\left(\frac{-1}{2}\right)^{n-1}.$$

Exercise 4.3. Consider the power series

$$\sum_{n=0}^{\infty} c_n x^n = \frac{1-2x^2}{1-5x+8x^2-4x^3} = 1 + 5x + 15x^2 + 39x^3 + \cdots$$

(a) Give a linear recurrence relation that (together with the initial conditions above) determines the sequence of coefficients $(c_n : n \geq 0)$ uniquely.

(b) Derive a formula for c_n as a function of $n \geq 0$.
[Hint: $1 - 5x + 8x^2 - 4x^3 = (1-x)(1-4x+4x^2)$.]

Exercise 4.4. Consider the power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{x+7x^2}{1-3x^2-2x^3}.$$

(a) Write down a linear recurrence relation and enough initial conditions to determine the sequence $(a_n : n \in \mathbb{N})$ uniquely.
(b) Given that $1 - 3x^2 - 2x^3 = (1-2x)(1+x)^2$, obtain a formula for a_n as a function of $n \in \mathbb{N}$.

Exercise 4.5. Consider the power series

$$\sum_{n=0}^{\infty} c_n x^n = \frac{3 - 11x + 11x^2}{1 - 4x + 5x^2 - 2x^3} = 3 + x + 0x^2 + x^3 + 6x^4 + 19x^5 + \cdots$$

(a) Give a linear recurrence relation that (together with the initial conditions above) determines the sequence of coefficients $(c_n : n \geq 0)$ uniquely.
(b) Derive a formula for c_n as a function of $n \geq 0$.

Exercise 4.6. A sequence of integers is determined by the initial conditions $g_0 = 1$, $g_1 = 2$, $g_2 = 3$, and the recurrence relation $g_n = 2g_{n-1} - g_{n-2} + 2g_{n-3}$ for all $n \geq 3$.

(a) Obtain a rational function formula for the generating series

$$G(x) = \sum_{n=0}^{\infty} g_n x^n = 1 + 2x + 3x^2 + 6x^3 + 13x^4 + 26x^5 + 51x^6 + \cdots .$$

(b) Obtain a formula for the coefficient g_n as a function of $n \in \mathbb{N}$.

Exercise 4.7. Define a sequence of numbers $(c_n : n \in \mathbb{N})$ by the initial conditions $c_0 = 1$, $c_1 = 2$, and $c_2 = 3$, and the recurrence relation $c_n = -c_{n-1} + 2c_{n-2} + 2c_{n-3}$ for all $n \geq 3$.

(a) Obtain an algebraic formula for the rational function

$$C(x) = \sum_{n=0}^{\infty} c_n x^n = 1 + 2x + 3x^3 + 3x^3 + 7x^4 + 5x^5 + \cdots .$$

(b) Obtain a formula for c_n as a function of $n \in \mathbb{N}$.

Exercise 4.8.

(a) Obtain a formula for the coefficients of the rational function

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = \frac{1+3x-x^2}{1-3x^2-2x^3}.$$

(b) Derive a recurrence relation and use it to check your answer.

Exercise 4.9. Define a sequence $(h_n : n \in \mathbb{N})$ by the initial conditions $h_0 = 1$, $h_1 = 2$, $h_2 = 0$, $h_3 = 5$, and the recurrence relation $h_n = -2h_{n-1} + h_{n-2} + 4h_{n-3} + 2h_{n-4}$ for all $n \geq 4$.

(a) Obtain an algebraic formula for the rational function

$$H(x) = \sum_{n=0}^{\infty} h_n x^n = 1 + 2x + 0x^2 + 5x^3 + 0x^4 + 9x^5 + \cdots .$$

(b) Obtain a formula for h_n as a function of $n \in \mathbb{N}$.

Exercise 4.10.

(a) Find rational numbers A, B, C such that for all $n \in \mathbb{N}$,

$$n^2 = A\binom{n+2}{2} + B(n+1) + C.$$

(b) Write $\sum_{n=0}^{\infty} n^2 x^n$ as a quotient of polynomials.

(c) Write $\sum_{n=0}^{\infty} n^3 x^n$ as a quotient of polynomials.

(d) For each $d \in \mathbb{N}$, let $F_d(x) = \sum_{n=0}^{\infty} n^d x^n$.
Show that $F_0(x) = 1/(1-x)$ and for all $d \geq 1$,

$$F_d(x) = x \frac{\mathrm{d}}{\mathrm{d}x} F_{d-1}(x).$$

(e) Let $F_d(x) = P_d(x)/(1-x)^{1+d}$. Derive a recurrence relation for the polynomials $P_d(x)$.

Exercise 4.11. Show that the converse of Theorem 4.14 holds. That is, assume that

$$g_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n + \cdots + p_s(n)\lambda_s^n$$

for all $n \geq N$, in which $p_i(n)$ is a polynomial of degree strictly less than d_i and the λ_i are distinct nonzero complex numbers, for $1 \leq i \leq s$. Let

$$Q(x) = (1-\lambda_1 x)^{d_1}(1-\lambda_2 x)^{d_2}\cdots(1-\lambda_s x)^{d_s}.$$

Then

$$G(x) = \sum_{n=0}^{\infty} g_n x^n = R(x) + \frac{P(x)}{Q(x)}$$

in which $P(x)$ and $R(x)$ are polynomials, and $\deg P(x) < \deg Q(x)$ and $\deg R(x) < N$.

Exercise 4.12.

(a) Show that $\dfrac{1}{\sqrt{1-4x}} = \displaystyle\sum_{k=0}^{\infty}\binom{2k}{k}x^k$.

(b) Deduce that for all $n \geq \mathbb{N}$, $\displaystyle\sum_{j=0}^{n}\binom{2j}{j}\binom{2n-2j}{n-j} = 4^n$.

(c)* Can you think of a combinatorial proof of part (b)?

Part II

Introduction to Graph Theory

Overview.

The idea of "six degrees of separation" is an enduring bit of popular culture. The claim is that for any two people on Earth – a school teacher in Kiribati and someone gardening in Punkeydoodles Corners, say – one can pass from the teacher to someone they have met, from them to a next acquaintance, and so on... arriving at the gardener in at most six steps. This is such an attractive idea that it doesn't really matter whether or not it is true. And there is no need to stick to the present time – how many steps would it take to get from you to Isaac Newton? Erdős numbers and the Kevin Bacon game are variations of the same idea.

Graph theory is the study of networks of this kind in the most abstract and general setting. The common structure is some set of things that are related to one another (or not) in pairs. Communication, transportation, ecological, and social networks can all be modeled in this way, and there are countless other applications.

In Chapter 5 we introduce graphs as mathematical structures and give a wide assortment of examples. We discuss a few elementary concepts of graph theory, the most important of which is that of isomorphism. This formalizes the intuitive notion of when two graphs are "basically the same".

In Chapter 6 we discuss walks in graphs, the concept of connectedness, and related results about substructures inside a graph. We also give Euler's solution to the "seven bridges of Königsberg" problem – arguably the first theorem of graph theory in history.

Chapter 7 builds upon Chapter 6, establishing some fundamental structural properties of graphs. These are important not only for the abstract theory, but also as a framework on which various practical algorithms are built.

In Chapter 8 we address the question of which graphs can be drawn in the plane without crossing edges – the answer is surprisingly recent, due to Kuratowski in 1930. The five Platonic solids – familiar from high-school geometry – also make an appearance.

In Chapter 9 we discuss colourings of graphs. The famous Four Colour Theorem takes center stage – ideas related to it have had a profound influence on the development of graph theory (and combinatorics, more generally) since the question was first posed in the 1850s.

Finally, Chapter 10 develops the theory of matchings, particularly in bipartite graphs. The main results are again surprisingly recent, due to König and to Hall in the 1930s. And again, they have had a strong influence on graph theory and optimization, right up to the present day.

In Chapters **??** and **??** we discuss a few other topics involving graphs, algorithms, optimization, or linear algebra.

Chapter 5

Graphs and Isomorphism.

5.1 Graphs.

Definition 5.1 (Graphs)**.** A *graph* $G = (V, E)$ is an ordered pair of sets in which the elements of E are 2-element subsets of V.

An element of the set V is called a *vertex*. The plural form of vertex is *vertices*. An element of E is called an *edge*. For a graph $G = (V, E)$ we also write $V(G) = V$ for the vertex-set and $E(G) = E$ for the edge-set. Unless mentioned otherwise, we only consider finite graphs.

Example 5.2. Here is a graph:

$$G = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\}).$$

This graph has vertex-set

$$V(G) = \{1, 2, 3, 4, 5\}$$

and edge-set

$$E(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\}.$$

That is mathematically precise, but difficult for a human being to grasp intuitively.

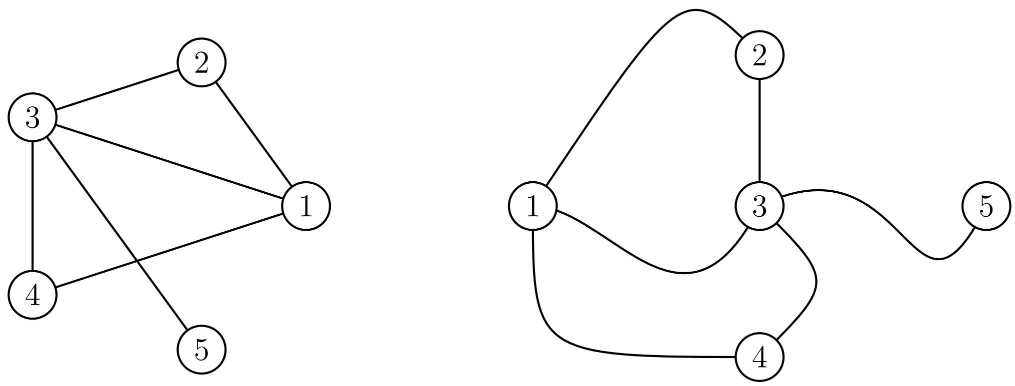


Figure 5.1: Two pictures of the graph in Example 5.2

We draw a picture of a graph $G = (V, E)$ by representing the vertices as dots, and joining two dots by a curve when the corresponding vertices are both elements of the same edge. Then, labelling the dots in the picture with the names of the vertices in V, we can see which pairs of vertices are joined by edges, and so determine the set of edges E, too. Figure 5.1 shows two different pictures of the graph in Example 5.2.

There is one *empty graph* $(\varnothing, \varnothing)$ with no vertices. For any finite set V of vertices there is one *edgeless graph* $(V, \varnothing)$ with no edges. (These are also often called "empty graphs" when the vertex-set is known.) For any finite set V of vertices there is one *complete graph* $K_V = (V, E)$ in which E is the set of all 2-element subsets of V. If $|V| = n$ then $|E(K_V)| = \binom{n}{2}$.

5.2 The Handshake Lemma.

Definition 5.3 (Adjacency, Incidence, Degree)**.**
Let $G = (V, E)$ be a graph.

1. Two vertices $v, w \in V$ are *adjacent*, or are *neighbours*, if $\{v, w\} \in E$ is an edge. We also use the notation vw for the edge $\{v, w\}$. The *neighbourhood* of $v \in V$ is the set

$$N(v) = \{w \in V : \; vw \in E\}$$

of vertices adjacent to V.

2. A vertex $v \in V$ and edge $e \in E$ are *incident* if $v \in e$. The two vertices incident with an edge $e \in E$ are the *ends* of e.
3. The *degree* of a vertex $v \in V$ is the number of edges incident with it. This number is denoted by $\deg(v)$. That is,

$$\deg(v) = |\{e \in E : \ v \in e\}| = |N(v)|.$$

4. The *degree sequence* of G is the multiset of degrees of the vertices of G. It is usually represented as a sequence of $|V|$ natural numbers sorted into weakly decreasing order.
5. A graph is *regular* if all vertices have the same degree. If all vertices have degree d then G is *d-regular*.

The degree sequence of the graph in Example 5.2 is $(4\,3\,2\,2\,1)$. A 0-regular graph is an edgeless graph. What does a 1-regular graph look like? What does a 2-regular graph look like? A 3-regular graph can be very very complicated.

Theorem 5.4 (The Handshake Lemma)**.** *Let $G = (V, E)$ be a graph. Then*

$$\sum_{v \in V} \deg(v) = 2 \cdot |E|.$$

Proof. We count the size of the set of pairs

$$X = \{(w, f) : \ w \in V \text{ and } f \in E \text{ and } w \in f\}$$

in two different ways. First, for every vertex $v \in V$ there are $\deg(v)$ pairs (w, f) in X with vertex v as the first coordinate. Therefore,

$$|X| = \sum_{v \in V} \deg(v).$$

Second, for every edge $e \in E$ there are 2 pairs (w, f) in X with edge e as the second coordinate. Therefore,

$$|X| = 2 \cdot |E|.$$

This proves the result. $\square$

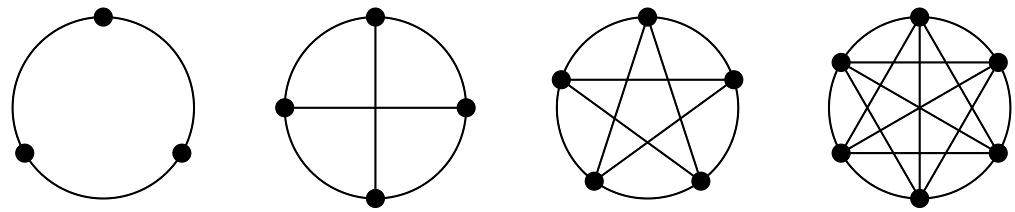


Figure 5.2: Complete graphs K_3, K_4, K_5, K_6.

Corollary 5.5. *Let $G = (V, E)$ be a graph. Then G has an even number of vertices of odd degree.*

Proof. Let S_0 be the set of vertices of even degree, and let S_1 be the set of vertices of odd degree. Consider the Handshake Lemma (modulo 2):

$$\begin{aligned} 0 \equiv 2 \cdot |E| \equiv \sum_{v \in V} \deg(v) &\equiv \sum_{v \in S_0} \deg(v) + \sum_{v \in S_1} \deg(v) \\ &\equiv \sum_{v \in S_0} 0 + \sum_{v \in S_1} 1 \equiv |S_1| \pmod 2. \end{aligned}$$

That proves the claim. □

5.3 Examples.

In this section we give some specific examples of graphs, as material to think about. The serious investigation of graph properties begins in Section 5.4.

Example 5.6 (Complete Graphs)**.** For $n \in \mathbb{N}$, the *complete graph* K_n has

$$\begin{aligned} \text{vertices} \quad & V(K_n) = \{1, 2, ..., n\} \\ \text{edges} \quad & E(K_n) = \{\{v, w\} : \ v, w \in V \text{ and } v \neq w\}. \end{aligned}$$

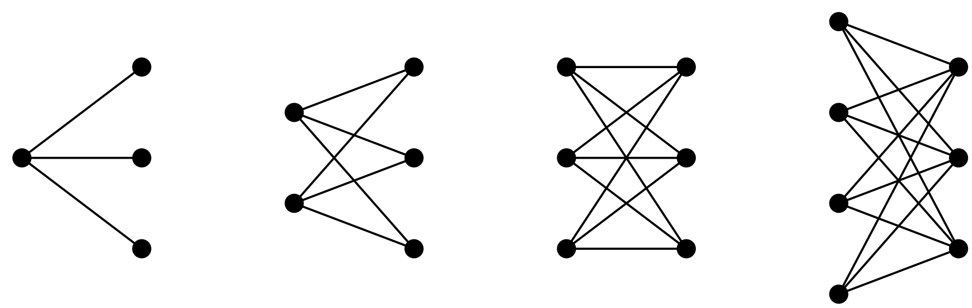


Figure 5.3: Complete bipartite graphs $K_{1,3}$, $K_{2,3}$, $K_{3,3}$, $K_{4,3}$.

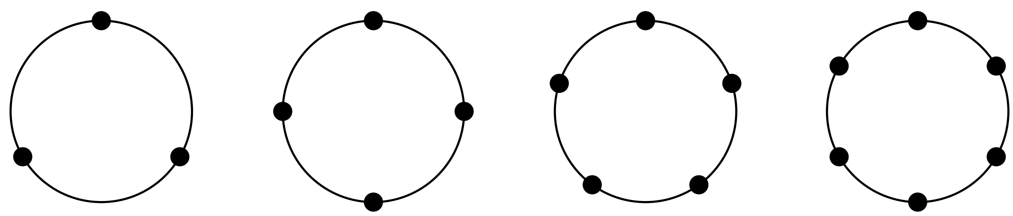


Figure 5.4: Cycles C_3, C_4, C_5, C_6.

Example 5.7 (Complete Bipartite Graphs)**.** For $r, s \in \mathbb{N}$, the *complete bipartite graph* $K_{r,s}$ has

$$\begin{aligned} \text{vertices} \quad & V(K_{r,s}) = \{a_1, a_2, ..., a_r\} \cup \{b_1, b_2, ..., b_s\} \\ \text{edges} \quad & E(K_{r,s}) = \{\{a_i, b_j\} : \ 1 \leq i \leq r \text{ and } 1 \leq j \leq s\}. \end{aligned}$$

Example 5.8 (Paths and Cycles)**.** For $n \geq 1$, the *path* P_n has

$$\begin{aligned} \text{vertices} \quad & V(P_n) = \{1, 2, ..., n\} \\ \text{edges} \quad & E(P_n) = \{\{i, i+1\} : \ 1 \leq i \leq n-1\}. \end{aligned}$$

For $n \geq 3$, the *cycle* C_n has

$$\begin{aligned} \text{vertices} \quad & V(C_n) = \{1, 2, ..., n\} \\ \text{edges} \quad & E(C_n) = \{\{i, i+1\} : \ 1 \leq i \leq n-1\} \cup \{\{1, n\}\}. \end{aligned}$$

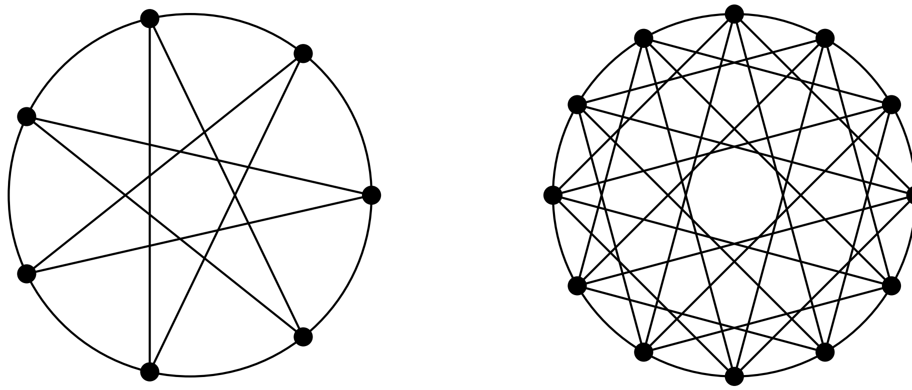


Figure 5.5: Circulants $C_7(1,3)$ and $C_{12}(1,3,5)$.

Example 5.9 (Circulants)**.** Fix $n \geq 2$ and let $\mathbb{Z}_n = \{[0],[1],...,[n-1]\}$ denote the integers (modulo n). Let S be any subset of $\mathbb{Z}_n$ with $[0] \notin S$. The *circulant* $C_n(S)$ has

$$\begin{aligned} \text{vertices} \quad & V(C_n(S)) = \mathbb{Z}_n \\ \text{edges} \quad & E(C_n(S)) = \{\{[v],[v+s]\} : [v] \in \mathbb{Z}_n \text{ and } [s] \in S\}. \end{aligned}$$

To simplify the notation, we write $C_{10}(1,3,4)$ instead of $C_{10}(\{[1],[3],[4]\})$, and so on. From the definition of $C_n(S)$, if $[a] \in S$ then it makes no difference whether or not $[-a] \in S$. In either case, the resulting graph will be the same.

Notice that the circulant $C_n(1)$ "looks the same as" the cycle C_n. Also, the circulant $C_n(\mathbb{Z}_n \setminus \{[0]\})$ "looks the same as" the complete graph K_n. Less obviously, the circulant $C_{12}(1,3,5)$ in Figure 5.5 "looks the same as" the complete bipartite graph $K_{6,6}$. This idea of two graphs looking the same is made precise in Section 5.4.

Example 5.10 (Hypercubes)**.** For $d \geq 0$, the *d-dimensional cube* Q_d has

$$\begin{aligned} \text{vertices} \quad & V(Q_d) = \{0,1\}^d \\ \text{edges} \quad & E(Q_d) = \{\{\alpha,\beta\} : |\alpha - \beta| = 1\}. \end{aligned}$$

In this definition, $|\alpha - \beta|$ is the distance between α and β as vectors in Euclidean space $\mathbb{R}^d$.

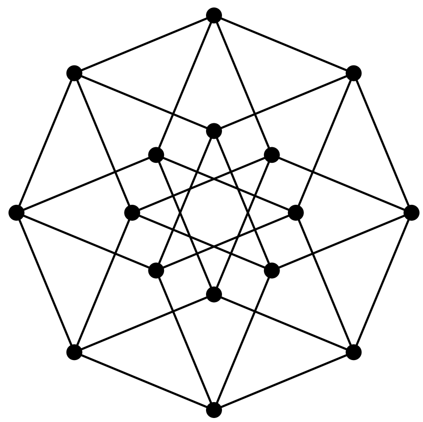


Figure 5.6: The four-dimensional cube Q_4.

Example 5.11 (Word Graphs)**.** For $\ell \geq 1$, the *word graph* $\mathrm{Word}(\ell)$ has vertex-set consisting of all ℓ-letter words in the English language. Two words are adjacent in $\mathrm{Word}(\ell)$ if and only if they differ by the substitution of exactly one letter in one position.

For instance, in $\mathrm{Word}(4)$ we can hop along the edges from `frog` to `toad` as follows.

$$\begin{array}{l} \texttt{frog} - \texttt{flog} - \texttt{flag} - \texttt{flat} - \texttt{feat} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad | \\ \texttt{toad} - \texttt{goad} - \texttt{goat} - \texttt{boat} - \texttt{beat} \end{array}$$

When I found this path in $\mathrm{Word}(4)$ I didn't know that "trog" (British slang for a stupid oafish person) and "trad" (traditional jazz or folk music) are actual English words. Here is a shorter path:

$$\texttt{frog} - \texttt{trog} - \texttt{trod} - \texttt{trad} - \texttt{toad}$$

Finding shortest paths between vertices in graphs is an important subject which we consider in Section **??**.

Example 5.12 (Unit Square Graphs)**.** Fix a positive real number $r > 0$. Choose n points in the unit square $[0,1] \times [0,1]$ independently and uniformly at random. Join two of these points by a line segment if they are

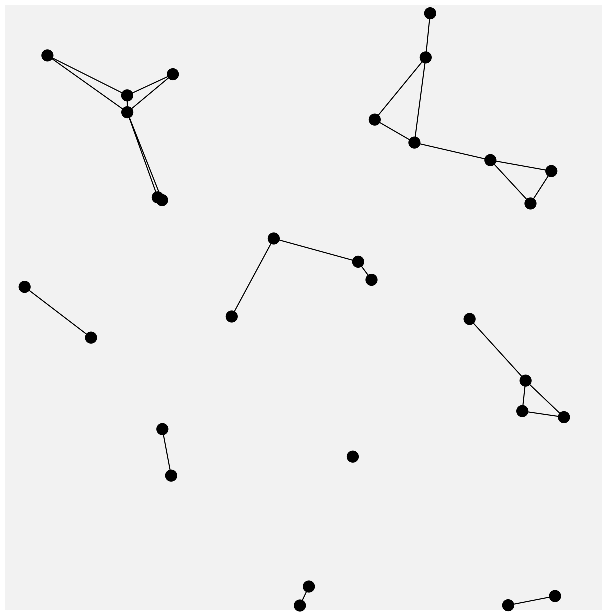


Figure 5.7: A unit square graph with $n = 30$ and $r \approx 0.179$.

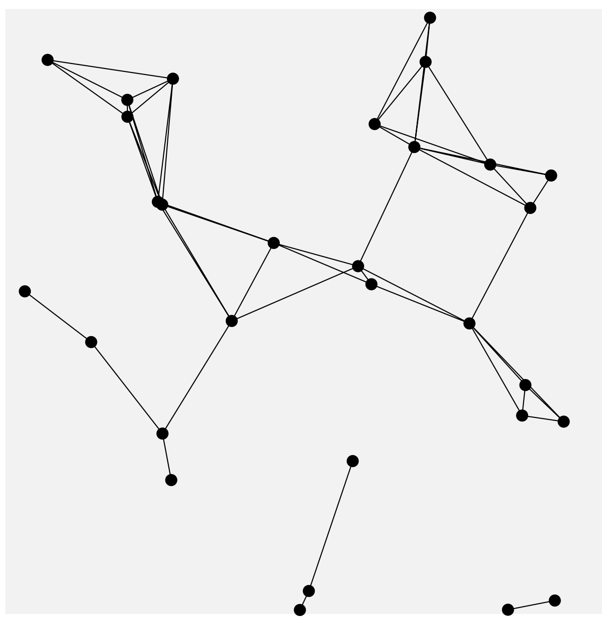


Figure 5.8: A unit square graph with $n = 30$ and $r \approx 0.232$.

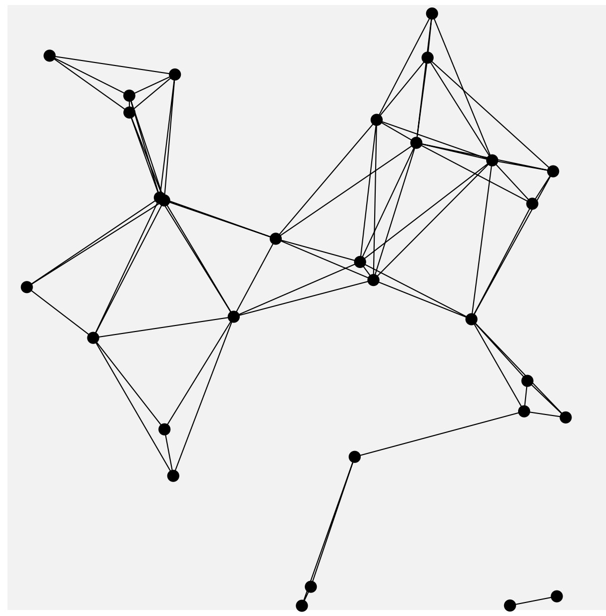


Figure 5.9: A unit square graph with $n = 30$ and $r \approx 0.291$.

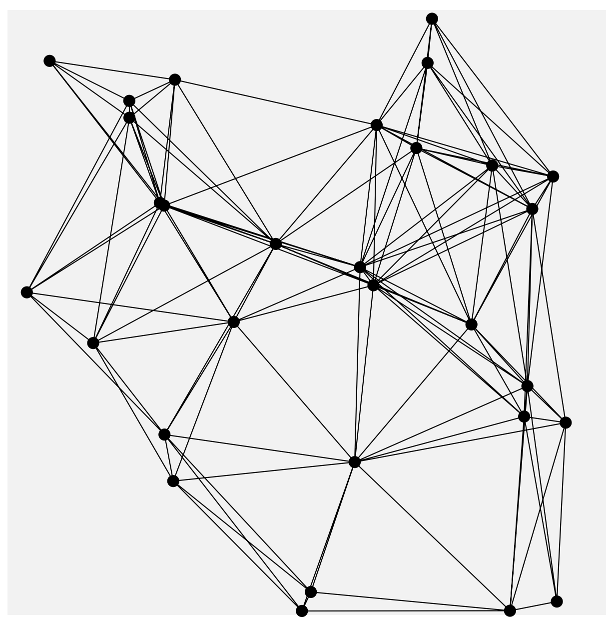


Figure 5.10: A unit square graph with $n = 30$ and $r \approx 0.378$.

within distance r of each other. These points and lines can be thought of as the vertices and edges of a graph.

Figures 5.7 to 5.10 give examples of unit square graphs. They all have the same randomly chosen vertex-set of $n = 30$ points, for four different values of $r > 0$.

5.4 Isomorphism.

Example 5.13. Consider the three graphs pictured in Figure 5.11. The graphs G and H are the same graph, since they have the same set of vertices $V(G) = V(H)$ and the same set of edges $E(G) = E(H)$. That is, $G = H$; these graphs are equal even though the pictures "look different". The graph J is not equal to the graph G because these graphs do not have the same set of vertices: $V(G) \neq V(J)$. But from the picture it is clear that J "looks the same as" G.

The concept of isomorphism makes this idea mathematically precise.

Definition 5.14 (Isomorphism)**.** Let G and H be graphs. An *isomorphism from G to H* is

- a bijection $f : V(G) \to V(H)$ from the vertices of G to the vertices of H, such that
- for all $v, w \in V(G)$,

$$\{f(v), f(w)\} \in E(H) \text{ if and only if } \{v, w\} \in E(G).$$

If there is an isomorphism from G to H then we say that *G is isomorphic to H* and write $G \simeq H$.

When $G \simeq H$, we also say that G is isomorphic with H, or that G and H are isomorphic. Informally, an isomorphism is a bijection between vertices that sends edges to edges and non-edges to non-edges. In other words, it is a bijection f such that both f and its inverse function f^{-1} preserve adjacency. We sometimes write $f : G \to H$ for an isomorphism $f : V(G) \to V(H)$ to simplify the notation.

The relation $\simeq$ of isomorphism is an equivalence relation, for the follow-

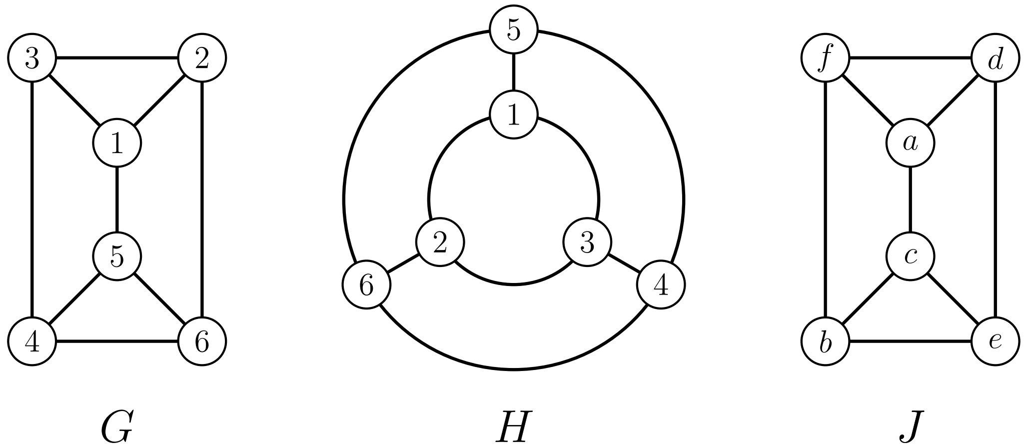


Figure 5.11: G equals H and is isomorphic with J.

ing reasons.

- For any graph G, the identity function $\iota : G \to G$ is an isomorphism, so that $G \simeq G$. This is the *reflexive* property. (Here, the identity function $\iota : V \to V$ is such that $\iota(v) = v$ for all $v \in V$.)
- If $f : G \to H$ is an isomorphism then $f^{-1} : H \to G$ is an isomorphism. Thus, if $G \simeq H$ then $H \simeq G$. This is the *symmetric* property.
- If $f : G \to H$ and $g : H \to J$ are isomorphisms, then $g \circ f : G \to J$ is an isomorphism. This is the *transitive* property.

Example 5.15.

- For the graphs in Figure 5.11, the following function $f : V(G) \to V(J)$ is an isomorphism.

v	1	2	3	4	5	6
$f(v)$	a	d	f	b	c	e

Here is another function $g : V(G) \to V(J)$ which is also an isomorphism.

v	1	2	3	4	5	6
$g(v)$	b	e	c	a	f	d

- For the isomorphisms f and g in the previous point, $g^{-1} \circ f : G \to G$

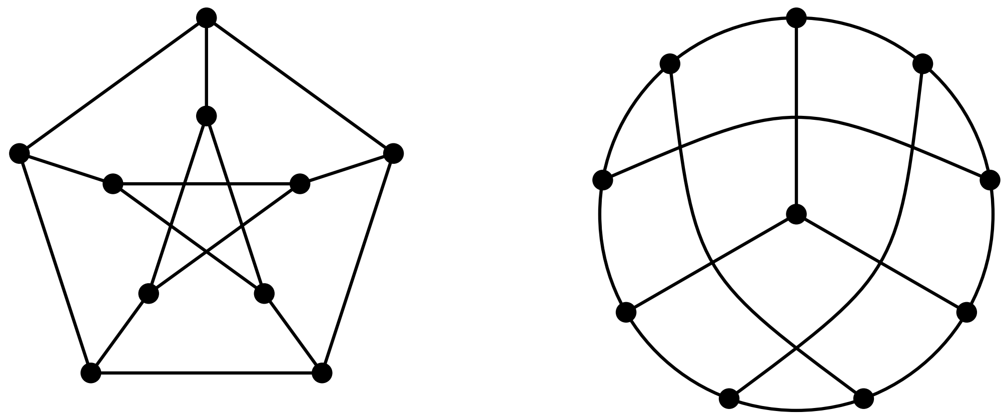


Figure 5.12: Two pictures of the Petersen graph.

is an isomorphism from G to G that is not the identity function. An isomorphism from a graph to itself is called an *automorphism*.

- For any circulant graph $C_n(S)$, the function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ defined by $f([a]) = [a+1]$ for all $[a] \in \mathbb{Z}_n$ is an automorphism of $C_n(S)$.
- The two graphs pictured in Figure 5.12 are isomorphic. To see this, find a way to label the vertices of each graph with the numbers $\{1, 2, ..., 10\}$ so that the edge-sets are the same. This is a famous graph known as the *Petersen graph*.
- For any $r \geq 1$, the circulant $C_{2r}(1, 3, 5, ..., 2r-1)$ is isomorphic to the complete bipartite graph $K_{r,r}$.

In order to show that two graphs G and H are isomorphic, it is enough to exhibit an isomorphism between them. Finding such an isomorphism might be a tricky problem, but given one particular bijection $f : V(G) \to V(H)$ it is not difficult to check whether or not f is an isomorphism.

In order to show that two graphs G and H are not isomorphic, one must show that none of the bijections between their vertex-sets can be an isomorphism. If G and H have n vertices, then there are $n!$ such bijections, and so checking them all individually is not a practical strategy. Instead, there are many necessary conditions that follow from the fact that $G \simeq H$. If any of these necessary conditions fails to hold, then we can conclude that G and H are not isomorphic. More generally, we can use these necessary conditions to narrow down our search for an isomorphism between two given graphs.

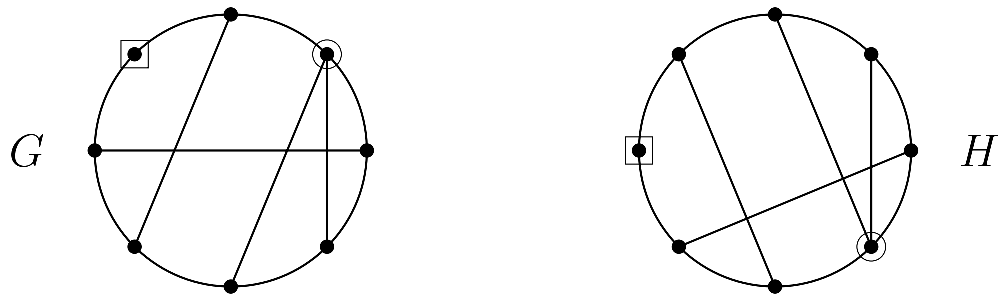


Figure 5.13: G is not isomorphic to H.

Lemma 5.16. *Let $f : G \to H$ be an isomorphism between graphs G and H.*

(a) For all $v \in V(G)$, $\deg_H(f(v)) = \deg_G(v)$.
(b) The graphs G and H have the same degree sequence.
(c) For all $v \in V(G)$, if $f(v) = w$ then the multiset of degrees of the neighbours of v in G is equal to the the multiset of degrees of the neighbours of w in H.

Proof. For part (a), since f is an isomorphism it restricts to a bijection from the set $N_G(v)$ of neighbours of v in G to the set $N_H(f(v))$ of neighbours of $f(v)$ in H. Therefore, $\deg_G(v) = |N_G(v)| = |N_H(f(v))| = \deg_H(f(v))$. Part (b) follows from part (a) since $f : V(G) \to V(H)$ is a bijection. Similarly, since f restricts to a bijection from $N_G(v)$ to $N_H(w)$, part (a) also implies part (c). □

Example 5.17. The two graphs pictured in Figure 5.13 have the same degree sequence ($4\,3\,3\,3\,3\,3\,3\,2$) but they are not isomorphic, for the following reason. They both have only one vertex of degree 4 (marked with a circle), and only one vertex of degree 2 (marked with a square). By Lemma 5.16(a), if $f : G \to H$ were an isomorphism then the circle would map to the circle, and the square would map to the square. But in G there is a vertex adjacent to both marked vertices, while in H there is no such vertex. Therefore, there is no isomorphism from G to H.

5.5 Some more Basic Concepts.

Definition 5.18 (Subgraphs)**.** Let $G = (V, E)$ be a graph.

1. A *subgraph* of G is a pair $H = (W, F)$ such that: W is a subset of V; F is a subset of E; and (W, F) is a graph.
2. A subgraph $H = (W, F)$ of G is *proper* if $H \neq (\varnothing, \varnothing)$ and $H \neq (V, E)$.
3. A subgraph $H = (W, F)$ of G is *spanning* if $W = V$. That is, H contains all the vertices of G.
4. A subgraph $H = (W, F)$ of G is *induced* if $F = \{e \in E : e \subseteq W\}$. That is, H contains all the edges of G that have both ends in W. We say that this is the *subgraph of G induced by $W \subseteq V$*, and denote it by $G[W]$ or $G|_W$.
5. If $S \subseteq E$ is a set of edges of G then the *deletion of S from G* is the spanning subgraph $G \smallsetminus S = (V, E \smallsetminus S)$.
6. If $S \subseteq V$ is a set of vertices of G then the *deletion of S from G* is the subgraph $G[V \smallsetminus S]$ of G induced by $V \smallsetminus S$.

The point of the last condition in Definition 5.18.1 (that (W, F) is a graph) is that if one chooses an edge e of G to be in the subgraph H, then one must also choose both of the vertices at the ends of e to be in H as well. For edge-deletion, we also write $G \smallsetminus e$ instead of $G \smallsetminus \{e\}$ to simplify notation. For vertex-deletion, we also use the notation $G \smallsetminus S$ for $G[V \smallsetminus S]$, and $G \smallsetminus v$ for $G \smallsetminus \{v\}$.

The empty graph $(\varnothing, \varnothing)$ is a subgraph of every graph. Every graph G is a spanning subgraph of the complete graph $K_{V(G)}$ with the same vertex-set. To get a feel for these concepts, consider the following questions. What does a proper spanning subgraph look like? What does a proper induced subgraph look like? What does an induced spanning subgraph look like?

Informally, we often say that "G has H as a subgraph" or "G contains H" to mean that there is a subgraph of G that is isomorphic with H. This can also be qualified with the adjectives "proper", "spanning", or "induced". For instance, the 4-cube Q_4 in Figure 5.6 has C_{16} as a proper spanning subgraph. (It is not too hard to find one.) A spanning cycle in a graph is called a *Hamilton cycle*.

Lemma 5.19. *Let $f : G \to H$ be an isomorphism between graphs G and H.*

(a) For any $v \in V$, if $f(v) = w$ then the subgraph of G induced by the neighbours of v is isomorphic to the subgraph of H induced by the neighbours of w. That is, $G[N_G(v)] \simeq H[N_H(w)]$.
(b) For any $d \in \mathbb{N}$, the subgraph of G induced by vertices of degree d is isomorphic to the subgraph of H induced by vertices of degree d.
(c) For any graph J, the number of induced subgraphs of G that are isomorphic to J is equal to the number of induced subgraphs of H that are isomorphic to J.

Proof. Exercise 5.10. □

Example 5.20. Revisiting the graphs pictured in Figure 5.13, let G' be the subgraph of G induced by its vertices of degree 3, and let H' be the subgraph of H induced by its vertices of degree 3. One sees that G' and H' do not have the same degree sequence. Lemma 5.16(b) implies that $G' \not\simeq H'$. Finally, Lemma 5.19(b) implies that $G \not\simeq H$.

Definition 5.21 (Graph Complement)**.** Let $G = (V, E)$ be a graph. Let $\mathcal{B}(V, 2)$ be the set of all 2-element subsets of V. The *complement* of G is the graph $G^c = (V, E^c)$ in which $E^c = \mathcal{B}(V, 2) \smallsetminus E$.

$$\begin{aligned} \text{vertices} \quad & V(G^c) = V(G) \\ \text{edges} \quad & E(G^c) = \{vw : \ vw \notin E(G)\}. \end{aligned}$$

The graph G^c has the same set of vertices as G, and vertices v and w are adjacent in G^c if and only if v and w are not adjacent in G. This could also be described as $G^c = K_{V(G)} \smallsetminus E(G)$.

Definition 5.22 (Cartesian Product)**.** Let G and H be graphs. The *Cartesian product* $G \,\square\, H$ has vertices $V(G \,\square\, H) = V(G) \times V(H)$. Two vertices (v_1, v_2) and (w_1, w_2) are adjacent in $G \,\square\, H$ if and only if: either $v_1 = w_1$ in $V(G)$ and $v_2 w_2 \in E(H)$, or $v_1 w_1 \in E(G)$ and $v_2 = w_2$ in $V(H)$.

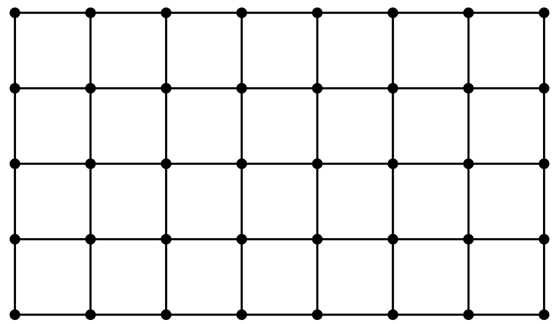


Figure 5.14: The grid $P_5 \,\square\, P_8$.

Example 5.23.

- For $r, s \geq 1$, the *r-by-s grid* is $P_r \,\square\, P_s$. The points and lines on a go board form the grid $P_{19} \,\square\, P_{19}$.
- The game board for nine-man morris (Figure 5.15) is a spanning subgraph of $C_8 \,\square\, P_3$.
- The d-dimensional cube Q_d of Example 5.10 is isomorphic to the Cartesian power $(K_2)^{\square d} = K_2 \,\square\, K_2 \,\square\, \cdots \,\square\, K_2$ of K_2 with d factors.
- For $r, d \in \mathbb{N}$, the *Hamming graph* is $H(r, d) = (K_r)^{\square d}$. Considering the 64 squares of a chessboard as vertices, with two vertices joined by an edge when they are one rook's move apart – that is, either in the same rank (row) or same file (column) – we have the Hamming graph $H(8, 2) = K_8 \,\square\, K_8$.
- For $\ell \geq 1$, the word graph $\mathrm{Word}(\ell)$ is an induced subgraph of the Hamming graph $H(26, \ell)$.

Definition 5.24 (Bipartite Graph)**.** A *bipartition* of a graph $G = (V, E)$ is an ordered pair (A, B) of subsets of V with the following two properties:

- both $A \cup B = V$ and $A \cap B = \varnothing$; and
- for every edge $e \in E$, both $e \cap A \neq \varnothing$ and $e \cap B \neq \varnothing$.

If a graph has a bipartition then it is a *bipartite* graph.

That is, for a bipartition (A, B) of a graph, every vertex is in exactly one of A or B, and every edge has one end in A and one end in B.

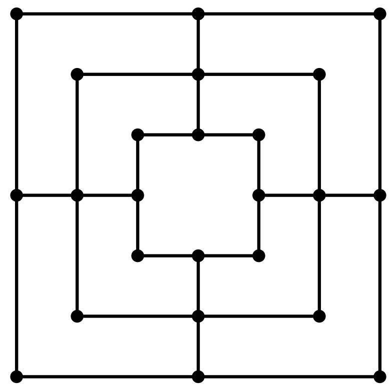


Figure 5.15: The game board for nine-man morris.

Exercise 5.5(a) gives a "bipartite version" of the Handshake Lemma.

Proposition 5.25. *Let G and H be graphs.*

(a) If $G \simeq H$ then G is bipartite if and only if H is bipartite.
(b) If G is bipartite then every subgraph of G is bipartite.
(c) For $n \geq 3$, the cycle C_n is bipartite if and only if n is even.
(d) If G contains an odd cycle then G is not bipartite.

Proof. For part (a), if $f : G \to H$ is an isomorphism and (A, B) is a bipartition of G, then $(\{f(v) : v \in A\}, \{f(w) : w \in B\})$ is a bipartition of H. This reasoning applies to f^{-1} as well, implying the result.

For part (b), let (A, B) be a bipartition of G. Let $H = (W, F)$ be a subgraph of G. Then $(W \cap A, W \cap B)$ is a bipartition of H, so that H is bipartite.

For part (c), regard C_n with the vertices and edges labelled as in Example 5.8. Assume that (A, B) is a bipartition of C_n. Switching the sets A and B if necessary, we may assume that $1 \in A$. Since $\{1, 2\}$ is an edge, it follows that $2 \in B$. Since $\{2, 3\}$ is an edge, it follows that $3 \in A$. By induction on j, it follows that for all $1 \leq j \leq n$, $j \in A$ if j is odd, and $j \in B$ if j is even. Thus, there is at most one possibility for a bipartition of C_n (with $1 \in A$). Only the edge $\{1, n\}$ has not been considered. If n is odd then this edge has both ends in A, and so C_n does not have a bipartition. If n is even, then the sets $A = \{1, 3, 5, ..., n - 1\}$ and $B = \{2, 4, 6, ..., n\}$ do form a bipartition of C_n.

Part (d) follows immediately from parts (b) and (c). $\square$

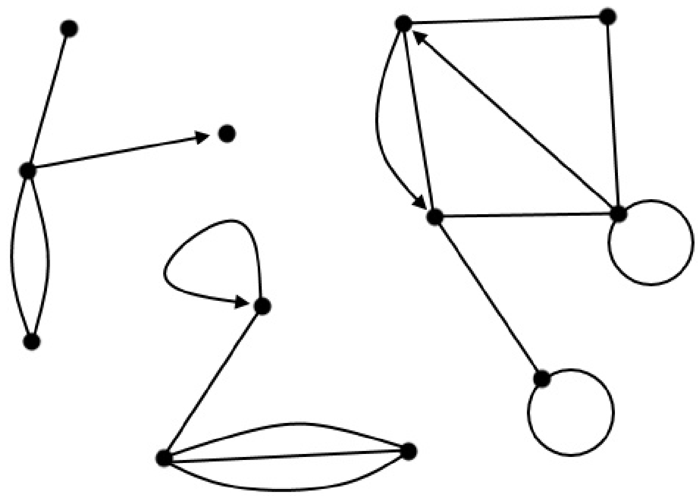


Figure 5.16: A general (mixed) graph.

The converse to Proposition 5.25(d) is also true. We could prove it now, but it will be much easier once we have developed the ideas in Chapter 7 (see Theorem 7.10).

5.6 Multigraphs and Directed Graphs.

For some purposes we would like to allow a graph to have more than one edge between a given pair of vertices, or to have an edge with both ends at the same vertex, or to have a direction on an edge from one end to the other. (Chapters 8 and **??** are two such situations.) See Figure 5.16 for a picture of such a *general graph*. We will not give a formal definition that covers all possibilities – in practice, different people use different definitions which are chosen for their convenience in a given context.

We will say that an *(undirected) multigraph* is a triple $G = (V, E, B)$ in which V is a set of *vertices*, E is a set of *edges*, and $B : V \times E \to \{0, 1, 2\}$ is an *incidence function*. The interpretation is that $B(v, e)$ is the number of ends of the edge e that are incident at the vertex v. We require that every edge $e \in E$ has exactly two ends (which may be equal) – that is, $\sum_{v\in V} B(v, e) = 2$. A *loop* at $v \in V$ is an edge $e \in E$ such that $B(v, e) = 2$. We use the notation $vew \in E(G)$ to indicate that the edge e is incident with vertices v and w. Thus, vew is a loop if and only if $v = w$. The *degree* of a vertex $v \in V$ is $\deg(v) = \sum_{e \in E} B(v, e)$. It is easy to see that the Handshake Lemma holds for multigraphs.

The *simplification* of a multigraph $G = (V, E, B)$ is a simple graph $\mathrm{si}(G)$ defined as follows. The vertex set is $V(G)$, and the edges are all pairs $\{v, w\}$ of vertices such that $B(v, e) = B(w, e) = 1$ for some $e \in E$. Informally, the simplification is obtained by removing all loops and replacing multiple edges between a pair of vertices by a single edge. The simplification of G can naturally be regarded as a spanning subgraph of G.

We will also say that a *directed multigraph* is a quadruple $G = (V, A, o, t)$ in which V is a set of *vertices*, A is a set of *arcs* (directed edges), $o : A \to V$ is an *origin function*, and $t : A \to V$ is a *terminus function*. The interpretation is that an arc $a \in A$ is directed out of $o(a)$ and into $t(a)$. The *indegree* of a vertex $v \in V$ is $\deg^+(v) = |t^{-1}(v)|$, the number of arcs pointing into v; its *outdegree* is $\deg^-(v) = |o^{-1}(v)|$, the number of arcs pointing out. The analogue of the Handshake Lemma for directed multigraphs is that

$$\sum_{v \in V} \deg^+(v) \;=\; |A| \;=\; \sum_{v \in V} \deg^-(v).$$

These definitions will suffice for our purposes. They could be combined to represent a *mixed multigraph* which contains both undirected edges and directed arcs. However, the vast majority of these notes involve only the *simple graphs* of Definition 5.1.

5.7 Exercises.

Exercise 5.1.

(a) Find a path from `hard` to `easy` in $\mathrm{Word}(4)$.
(b) Find a path from `wrong` to `right` in $\mathrm{Word}(5)$.

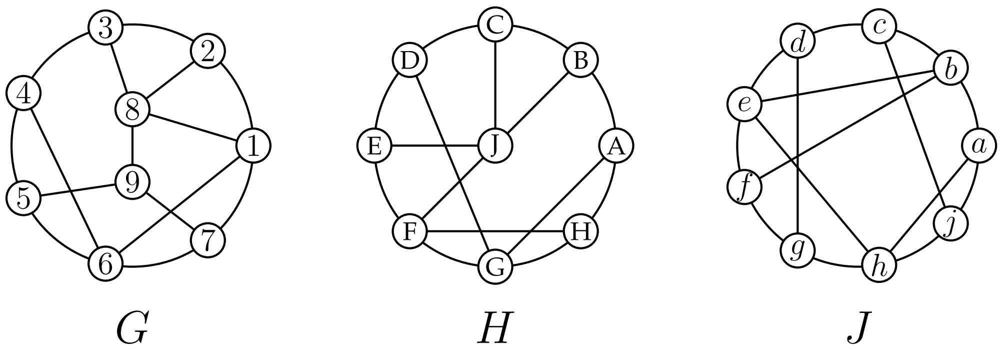


Figure 5.17: Three graphs for Exercise 5.6.

Exercise 5.2. Let G be a graph with at least two vertices. Show that G has two different vertices of the same degree.

Exercise 5.3. Assume that all vertices of $G = (V, E)$ have degree either 1 or 3. Show that if $|V| = |E|$ then the number of vertices of degree 1 is equal to the number of vertices of degree 3.

Exercise 5.4. Let $G = (V, E)$ be a graph, $S \subseteq V$ a subset of vertices, and $\partial S \subseteq E$ the set of edges that have exactly one end in S. Show that if every vertex in S has even degree, then $|\partial S|$ is even.

Exercise 5.5 (Bipartite Handshake Lemma)**.** Let $G = (V, E)$ be a graph with a bipartition (A, B).

(a) Show that $\sum_{a \in A} \deg(a) = |E| = \sum_{b \in B} \deg(b)$.

(b) Show that if G is regular with at least one edge, then $|A| = |B|$.

(c) Use part (b) to give another proof of Proposition 5.25(c).

Exercise 5.6. For each pair of the graphs pictured in Figure 5.17, give either an isomorphism between them or a reason why they are not isomorphic to each other.

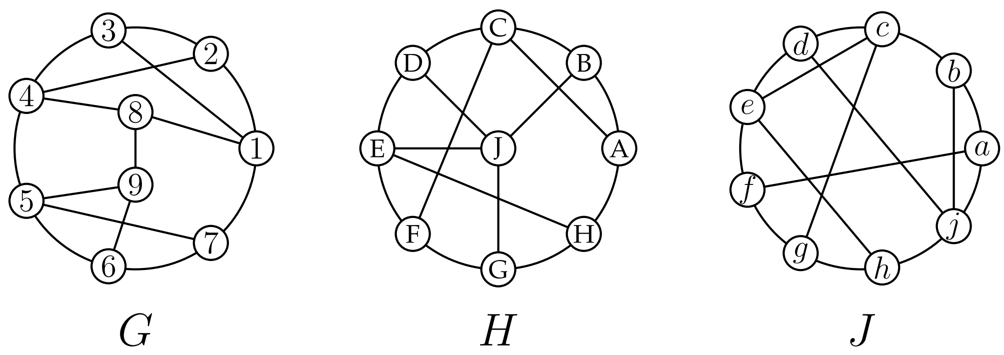


Figure 5.18: Three graphs for Exercise 5.7.

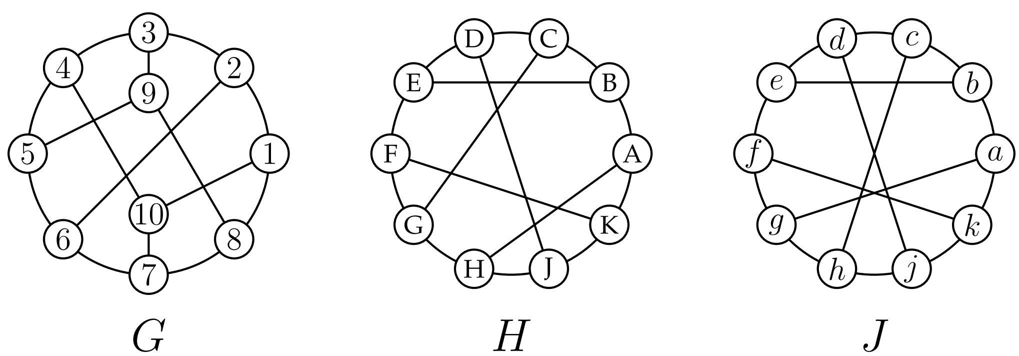


Figure 5.19: Three graphs for Exercise 5.9.

Exercise 5.7. For each pair of the graphs pictured in Figure 5.18, give either an isomorphism between them or a reason why they are not isomorphic to each other.

Exercise 5.8. Are any of the graphs in Figure 5.17 isomorphic to any of the graphs in Figure 5.18?

Exercise 5.9. For each pair of the graphs pictured in Figure 5.19, give either an isomorphism between them or a reason why they are not isomorphic to each other.

Exercise 5.10. Prove Lemma 5.19.

Exercise 5.11.

(a) There are two 3-regular graphs with 6 vertices (up to isomorphism). Draw pictures of them and explain why they are not isomorphic, and why any 6-vertex 3-regular graph is isomorphic to one or the other.

(b) There are several 3-regular graphs with 8 vertices (up to isomorphism). Draw pictures of them and explain why they are not isomorphic, and why any 8-vertex 3-regular graph is isomorphic to one of the graphs on your list.

Exercise 5.12. Recall the circulant graphs from Example 5.9.

(a) Draw pictures of $C_8(1,3)$, $C_8(1,5)$, and $C_9(1,3)$.

(b) Show that $C_{10}(2,3)$ and $C_{10}(1,4)$ and $C_{10}(1,6)$ are isomorphic to each other.

(c) Show that $C_{14}(1,3)$ is isomorphic to $C_{14}(1,5)$.

(d) Show that if $ab \equiv 1 \pmod{n}$, then $C_n(1,a)$ is isomorphic to $C_n(1,b)$.

Exercise 5.13. A graph G is *self-complementary* if it is isomorphic to its complement: $G \simeq G^{\mathsf{c}}$.

(a) If G is self-complementary then $|V(G)| \equiv 0$ or $1 \pmod{4}$.

(b) For each $k \in \mathbb{N}$, give an example of a self-complementary graph with $4k$ vertices.

(c) For each $k \in \mathbb{N}$, give an example of a self-complementary graph with $4k+1$ vertices.

(d) If G is a self-complementary graph with $4k+1$ vertices, then there is a $v \in V(G)$ such that $G \smallsetminus v$ is self-complementary.
[Hint: Let $f : G \to G^{\mathsf{c}}$ be an isomorphism, regarded as a permutation of $V(G)$. One of the cycles of this permutation must have odd length.]

Exercise 5.14. Let $G = (V, E)$ be a self-complementary graph with $4k$ vertices, and let $f : G \to G^{\mathsf{c}}$ be an isomorphism. Prove the following.

(a) For each $v \in V$, exactly one of v or $f(v)$ has degree (in G) strictly less than $2k$.
(b) Exactly $2k$ vertices of G have degree strictly less than $2k$ (in G).
(c) Define a new graph $G^+ = (W, F)$ by adding a new vertex $W = V \cup \{z\}$ and new edges zv for all vertices $v \in V$ that have degree strictly less than $2k$ in G. Show that G^+ is self-complementary.

Exercise 5.15. Let $G = (V, E)$ be a bipartite graph with $|V| = n$ vertices. Show that G has at most $|E| \leq \lfloor n^2/4 \rfloor$ edges.

Exercise 5.16. For each $k \geq 2$ and $n \geq 2k$, determine whether or not the circulant $C_n(1, k)$ is bipartite. Give a proof. [Hint: look at some small examples, then make a conjecture, then prove your conjecture.]

Exercise 5.17. Show that for any graphs G and H, the Cartesian product $G \,\square\, H$ is bipartite if and only if both G and H are bipartite.

Exercise 5.18. Let's say that a graph with a Hamilton (spanning) cycle is *Hamiltonian*. (A graph with a spanning path is *weakly Hamiltonian*.)

(a) Show that if G is bipartite and Hamiltonian then $|V(G)|$ is even.
(b) For which $r, s \geq 2$ is the grid $P_r \,\square\, P_s$ Hamiltonian? Explain.
(c) Show that for $n \geq 3$, the product $C_n \,\square\, K_2$ is Hamiltonian.
(d) Deduce that for $d \geq 2$, the d-cube Q_d is Hamiltonian. (Hamilton cycles in hypercubes are called "binary Gray codes".)

Exercise 5.19. For each $r \geq 1$ and $s \geq 3$, determine whether or not $K_{1,r} \,\square\, C_s$ is Hamiltonian. Give a proof. (Hint: look at some small examples, then make a conjecture, then prove your conjecture.)

Exercise 5.20. Is the Petersen graph Hamiltonian? Explain.

Exercise 5.21. Let $G = (V, E)$ be a graph. The *line-graph* $L(G)$ of G has the edges of G as its vertices, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G have a common end-vertex. That is,

$$\begin{aligned} \text{vertices} \quad & V(L(G)) = E(G) \\ \text{edges} \quad & E(L(G)) = \{ef : \; e \cap f \neq \varnothing\}. \end{aligned}$$

(a) Find a graph G that is isomorphic to its own line-graph.
(b) Find two graphs G and H such that $G \not\simeq H$, but $L(G) \simeq L(H)$.
(c) Show that for $r, s \in \mathbb{N}$, the product $K_r \,\square\, K_s$ is isomorphic to the line-graph $L(K_{r,s})$.

Exercise 5.22. For $d \geq 3$, the *d-th Odd graph* $\mathbf{O}_d$ is defined as follows. The vertices of $\mathbf{O}_d$ are the $(d-1)$-element subsets of the set $\{1, 2, ..., 2d-1\}$. Two vertices S and S' of $\mathbf{O}_d$ are adjacent if and only if $S \cap S' = \varnothing$. Prove the following claims.

(a) The Odd graph $\mathbf{O}_d$ is d-regular. How many vertices and edges does it have?
(b) The Petersen graph is isomorphic to $\mathbf{O}_3 = L(K_5)^{\mathsf{c}}$, the complement of the line-graph of K_5.
(c) The Odd graph $\mathbf{O}_d$ does not contain C_3 as a subgraph.
(d) The Odd graph $\mathbf{O}_d$ does have C_6 as an induced subgraph.
(e) The Odd graph $\mathbf{O}_d$ does not contain C_4 as a subgraph.
(f) The Odd graph $\mathbf{O}_d$ does have C_{2d-1} as an induced subgraph.

Exercise 5.23. Find an example of a graph $G = (V, E)$ and two vertices $v, w \in V$ with the following properties:

- the subgraphs $G \setminus v$ and $G \setminus w$ are isomorphic; but
- there is no automorphism $f : G \to G$ such that $f(v) = w$.

Chapter 6

Walks, Paths, and Connectedness.

6.1 Walks, Trails, Paths, and Cycles.

Definition 6.1 (Walks,Trails,Paths,Cycles)**.** Let $G = (V, E)$ be a graph.

1. A *walk* in G is a sequence of vertices $W = (v_0\, v_1 \ldots v_k)$ such that $v_{i-1}v_i \in E$ is an edge for all $1 \leq i \leq k$. Each consecutive pair $v_{i-1}v_i$ is a *step* of the walk. The *length* of the walk is $\ell(W) = k$, the number of steps.
2. The vertices v_0 and v_k are the *ends* of W, and W is a *walk from* v_0 *to* v_k, or a (v_0, v_k)-*walk*. The walk is *closed* if $v_0 = v_k$.
3. Two walks $W = (v_0\, v_1 \ldots v_k)$ and $Z = (z_0\, z_1 \ldots z_\ell)$ are equal if and only if $k = \ell$ and $v_i = z_i$ for all $0 \leq i \leq k$.
4. A walk W in G is a *trail* if the edges $v_{i-1}v_i$ are all different: if $1 \leq i < j \leq k$ then $\{v_{i-1}, v_i\} \neq \{v_{j-1}, v_j\}$.
5. A walk W in G is a *path* if the vertices v_i are all different: if $0 \leq i < j \leq k$ then $v_i \neq v_j$.
6. A walk W in G is a *cycle* if it is closed, the length is at least three, and the vertices v_i are all different except that $v_0 = v_k$. That is: $v_0 = v_k$, and $\ell(W) \geq 3$, and if $0 \leq i < j \leq k$ are such that $v_i = v_j$, then $i = 0$ and $j = k$.

A walk $W = (v_0\, v_1\, ...\, v_k)$ in G is *supported on* a particular subgraph of G: the one with

$$\begin{aligned} \text{vertex-set} \qquad & V(W) = \{v_0, v_1, ..., v_k\} \\ \text{and edge-set} \qquad & E(W) = \{v_0v_1, v_1v_2, ..., v_{k-1}v_k\}. \end{aligned}$$

The words "path" and "cycle" have been used before, for the graphs in Example 5.8. In Sections 5.4 and 5.5, we broadened the use of these words to mean any graph isomorphic to some P_n or C_n. A walk that is a path as in Definition 6.1 is supported on a path, and a walk that is a cycle as in Definition 6.1 is supported on a cycle.

In a multigraph, to specify a walk one must also specify the choice of edge between successive vertices at each step: $W = (v_0\, e_1\, v_1\, e_2\, v_2\, ...\, e_k\, v_k)$. If $e \neq f$ then $(v\, e\, w\, f\, v)$ is a cycle of length two, and a loop $(v\, e\, v)$ is a cycle of length one.

Definition 6.2 (Concatenation of walks)**.** Let $W = (v_0\, v_1\, ...\, v_k)$ and $Z = (z_0\, z_1\, ...\, z_\ell)$ be walks in a graph $G = (V, E)$. The *concatenation of W followed by Z* is defined only when $v_k = z_0$, in which case it is the walk

$$WZ = (v_0\, v_1\, ...\, v_{k-1}\, v_k\, z_1\, ...\, z_\ell)$$

obtained by superimposing the vertices $v_k = z_0$ and concatenating the sequences of steps.

Notice that for concatenation of walks, the length (number of steps) is additive: $\ell(WZ) = \ell(W) + \ell(Z)$.

Example 6.3. Consider the graph pictured in Figure 6.1.

- The sequence (`x a p r b c b q`) is not a walk, because `rb` is not an edge.
- The walk (`s q r d s z y r p`) is a trail, but not a closed trail.
- The closed walk (`r c d s q b c r`) is not a trail.
- The trail (`x q p r q s r c d s z`) is not a path.
- The closed walk (`y z y`) is not a cycle, and not a trail.
- The walk (`a b c b c d`) is not a path, although it is supported on a path.

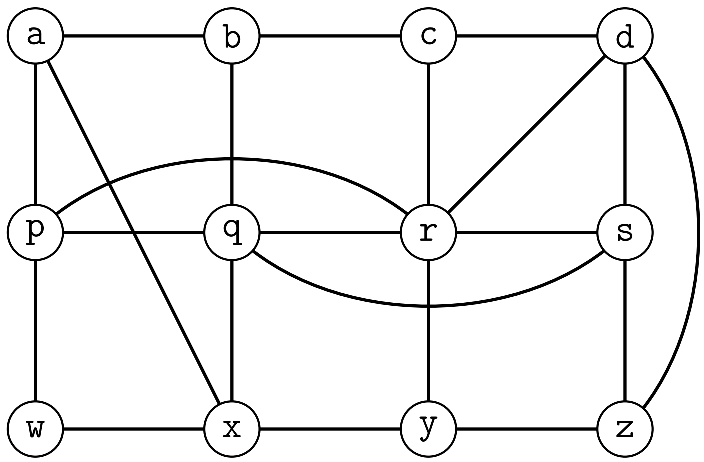


Figure 6.1: Graph for Example 6.3.

- The concatenation of the walks $W = (\mathtt{c\,r\,s\,q\,r})$ and $Z = (\mathtt{r\,s\,z\,d})$ is $WZ = (\mathtt{c\,r\,s\,q\,r\,s\,z\,d})$.
- The concatenation of the paths $P = (\mathtt{p\,q\,x\,a})$ and $Q = (\mathtt{p\,r\,q\,b})$ is not defined. However, $P\,(\mathtt{a\,p})\,Q = (\mathtt{p\,q\,x\,a\,p\,r\,q\,b})$.

Proposition 6.4. *Let W be a walk in a graph $G = (V, E)$. If W is either a path or a cycle then W is a trail.*

Proof. We prove the contrapositive form of this: if W is not a trail then it is not a path or a cycle. Let $W = (v_0\, v_1 \ldots v_k)$ be a walk in G, and assume that W is not a trail. Then W has a repeated edge: there exist indices $1 \le i < j \le k$ such that $\{v_{i-1}, v_i\} = \{v_{j-1}, v_j\}$. Now either $v_i = v_{j-1}$ or $v_i = v_j$. Since $1 \le i < j \le k$, if $v_i = v_j$ then this is a repeated vertex other than $v_0 = v_k$. This implies that W is neither a path nor a cycle. In the remaining case, $v_i = v_{j-1}$. If $i < j - 1$ then this is a repeated vertex other than $v_0 = v_k$, so that again W is neither a path nor a cycle. Finally, we have the case that $i = j - 1$ so that $v_i = v_{j-1}$ is not a repeated vertex. But in this case $v_{i-1} = v_j$ is a repeated vertex (since $\{v_{i-1}, v_i\} = \{v_{j-1}, v_j\}$). It follows that W is not a path. If W is a cycle then $i - 1 = 0$ and $j = k$ since $v_0 = v_k$ are the only repeated vertices allowed. But since $i = j - 1$ this implies that $k = 2$, and so W can not be a cycle since it has length two. □

Theorem 6.5. *Let $G = (V, E)$ be a graph and let $v, w \in V$ be any two vertices. Any walk from v to w of minimum length is a path.*

Proof. If $v = w$ then (v) is a walk from v to w of length zero. Clearly this is the minimum possible, (v) is the only walk of length zero from v to w, and it is a path. So we can assume that $v \neq w$.

Let $W = (v_0\, v_1 \ldots v_k)$ be a walk in G from $v_0 = v$ to $v_k = w$, and assume that $\ell(W) = k$ is as small as possible among all (v, w)-walks in G. Arguing for a contradiction, suppose that W is not a path. Then W has a repeated vertex: there are indices $0 \leq i < j \leq k$ such that $v_i = v_j$. Now the concatenation of walks $(v_0\, v_1 \ldots v_i)$ and $(v_j\, v_{j+1} \ldots v_k)$ is a (v, w)-walk of length $i + (k - j) < \ell(W)$. This contradicts the hypothesis that W is as short as possible among all (v, w)-walks. This contradiction shows that W has no repeated vertices. Therefore, W is a path. $\square$

Corollary 6.6. *For vertices v, w of a graph G, if there is a (v, w)-walk in G then there is a (v, w)-path in G.*

Proof. This follows immediately from Theorem 6.5. $\square$

Proposition 6.7. *Let $G = (V, E)$ be a graph and let $v, w \in V$ be distinct vertices. If G contains two distinct paths from v to w then G contains a cycle.*

Proof. Let $P = (v_0\, v_1 \ldots v_k)$ and $Q = (z_0\, z_1 \ldots z_\ell)$ be distinct paths in G from $v = v_0 = z_0$ to $w = v_k = z_\ell$. Since $P \neq Q$ there is an index a with $0 \leq a < \min\{k, \ell\}$ such that $v_0 = z_0$, $v_1 = z_1$, ..., $v_a = z_a$, but $v_{a+1} \neq z_{a+1}$. Since $v_k = w$ is on the path Q, there is a smallest index b such that $a + 1 \leq b \leq k$ and v_b is on the path Q. Let $0 \leq c \leq \ell$ be the index such that $z_c = v_b$. This index c is determined uniquely because Q has no repeated vertices. If $0 \leq c \leq a$ then $v_c = z_c = v_b$ would be a repeated vertex of P. But P is a path, so that this does not happen. Therefore, $a + 1 \leq c \leq \ell$. Also, if $b = a + 1$ then $c \geq a + 2$, because otherwise $v_{a+1} = v_b = z_c = z_{a+1}$ would contradict the way the index a was determined.

The vertices $v_a, v_{a+1}, ..., v_b$ are pairwise distinct since P has no repeated vertices, and the only ones of these vertices that are also on Q are $v_a = z_a$ and $v_b = z_c$. The vertices $z_a, z_{a+1}, ..., z_{c-1}, z_c$ are pairwise distinct since Q has no repeated vertices. It follows that the closed walk

$$C = (v_a\, v_{a+1} \ldots v_b\, z_{c-1}\, z_{c-2} \ldots z_{a+1}\, z_a)$$

has no repeated vertices except that $v_a = z_a$. Since the length of C is $\ell(C) = (b-a) + (c-a) \geq 3$, it follows that C is a cycle in G. □

Proposition 6.8. *Let $G = (V, E)$ be a nonempty graph in which every vertex has degree at least 2. Then G contains a cycle.*

Proof. Let $P = (v_0\, v_1 \ldots v_k)$ be a path in G of maximum length. Since G has a vertex and this vertex has degree at least 2, the path P has length at least 2. Since the degree of v_0 is at least 2, it has a neighbour $w \neq v_1$ different from v_1. If w is not a vertex on P then $(w\, v_0\, v_1 \ldots v_k)$ is a path in G that is longer than P. We assumed that P is a path of maximum length in G, so this does not happen. Therefore, the vertex w is on P, and so $w = v_j$ for some $2 \leq j \leq k$. Now $C = (v_0\, v_1 \ldots v_j\, v_0)$ is a cycle in G. □

6.2 Connectedness.

Definition 6.9 (Reachability)**.** Let $G = (V, E)$ be a graph. For vertices $v, w \in V$ we say that v *reaches* w *in* G if there exists a walk in G from v to w.

Lemma 6.10. *For a graph $G = (V, E)$, reachability is an equivalence relation on the set V of vertices.*

Proof. The relation of reachability has the following three properties. Let $u, v, w \in V$ be vertices.

- It is *reflexive:* (v) is a walk, so that v reaches v.
- It is *symmetric:* if $(z_0\, z_1 \ldots z_k)$ is a (v, w)-walk then $(z_k\, z_{k-1} \ldots z_0)$ is a (w, v)-walk. Thus, if v reaches w then w reaches v.

- It is *transitive:* if $(z_0\, z_1 \dots z_k)$ is a (u, v)-walk and $(t_0\, t_1 \dots t_\ell)$ is a (v, w)-walk, then $(z_0\, z_1 \dots z_k\, t_1 \dots t_\ell)$ is a (u, w)-walk. Thus, if u reaches v and v reaches w, then u reaches w.

$\square$

Definition 6.11 (Connected components, and connectedness)**.** Let $G = (V, E)$ be a graph. Let U_1, U_2, ..., U_c be the equivalence classes of the reachability relation on V. (Recall that equivalence classes are nonempty, by definition.)

1. The *connected components* of G are the subgraphs $G_i = G[U_i]$ of G induced by these equivalence classes.
2. The number of connected components of G is denoted by $c(G)$.
3. The graph G is *connected* when $c(G) = 1$.

Note that the empty graph $(\varnothing, \varnothing)$ is not connected according to Definition 6.11, because it has no connected components at all. A graph is connected if and only if it has exactly one connected component.

At this point we can now describe paths and cycles structurally, or intrinsically, using the concept of connectedness. A *cycle* is a connected 2-regular graph. A *path* is a connected graph in which all vertices have degree at most 2, and which is not a cycle.

Lemma 6.12. *Let $G = (V, E)$ be a graph with connected components G_1, ..., G_c, and let $v, w \in V$ be vertices. If v reaches w in G then v reaches w in G_i for exactly one $1 \leq i \leq c$.*

Proof. If v reaches w in G then v and w are in the same equivalence class U_i of reachability in G, by definition. Since equivalence classes are pairwise disjoint, there is exactly one equivalence class U_i containing both v and w. If W is a (v, w)-walk in G, then any two vertices in W are equivalent by reachability in G, so that the vertices of W are all in the same equivalence class U_i. Since $G_i = G[U_i]$ is the subgraph of G induced by U_i, it follows that W is a walk in G_i, so that v reaches w in G_i. $\square$

Corollary 6.13. *The connected components of a graph are connected graphs.*

Proof. Let $G_i = G[U_i]$ be a connected component of a graph G. Since U_i is nonempty (by definition of equivalence classes), G_i has at least one connected component. Let $v, w \in U_i$ be vertices of G_i. By definition of U_i, v reaches w in G. By Lemma 6.12, it follows that v reaches w in G_i. This shows that any two vertices of G_i are equivalent by reachability in G_i. Thus G_i is connected. □

Proposition 6.14. *Let $G = (V, E)$ be a graph. The following are equivalent.*

(a) G is connected.
(b) G is nonempty and every vertex $v \in V$ is such that for all $w \in V$, there is a (v, w)-path in G.
(c) There is a vertex $v \in V$ such that for all $w \in V$, there is a (v, w)-path in G.

Proof. First, assume that G is connected. Then G is not the empty graph. Let $v \in V$ be any vertex. For any vertex $w \in V$, v reaches w. By Theorem 6.5 there is a (v, w)-path in G. This shows that (a) implies (b). That (b) implies (c) is clear. If (c) holds then let $v \in V$ be such a vertex of G. Since $V \neq \varnothing$, G is nonempty. Since v reaches every vertex of G, and reachability is an equivalence relation, G has exactly one connected component – that is, G is connected. □

Let $G = (V, E)$ be a graph. For a set $S \subseteq V$ of vertices, the *boundary of S* is the set

$$\partial S = \{e \in E : \ |e \cap S| = 1\}$$

of edges of G that have exactly one end in S.

Theorem 6.15. *Let $G = (V, E)$ be a nonempty graph. The following are equivalent:*

(a) The graph G is connected.
(b) For every proper subset $\varnothing \neq S \subset V$, the boundary $\partial S \neq \varnothing$ is not empty.

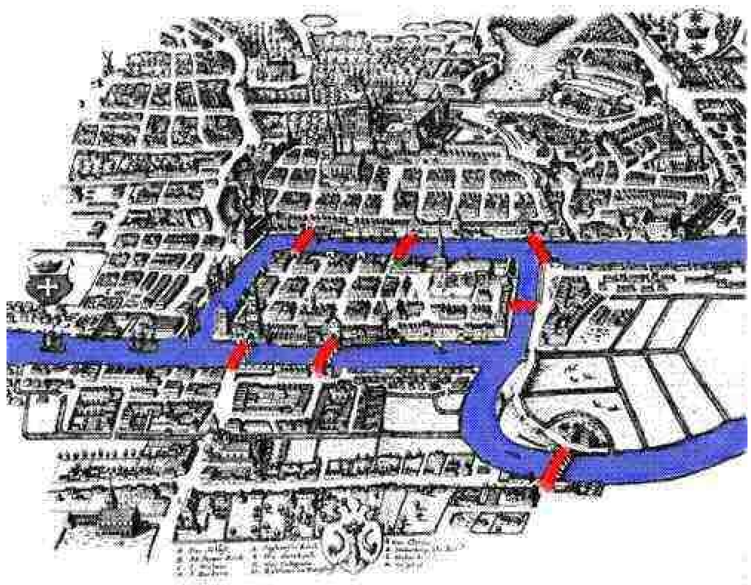


Figure 6.2: The seven bridges of Königsberg.

Proof. First, assume that G is connected and let S be a proper subset of V. Thus, there is a vertex $v \in S$ and a vertex $w \in V \smallsetminus S$. Since G is connected, there is a (v,w)-walk $W = (z_0\, z_1\, ...\, z_k)$ in G. Since $z_0 \in S$ and $z_k \notin S$, there is an index $1 \leq i \leq k$ such that $z_{i-1} \in S$ and $z_i \notin S$. Now $z_{i-1}z_i$ is an edge in ∂S, so that $\partial S \neq \varnothing$.

Second, assume that G is not connected. Since G is not empty, it has at least two connected components. Let S be the set of vertices of any connected component of G. This S is a proper subset of V. Suppose that $vw \in E$ has $v \in S$ and $w \notin S$. But v reaches w in G, so that $w \in S$. This contradiction shows that $\partial S = \varnothing$. □

6.3 Euler Tours.

In the 18th century, Königsberg was the easternmost major city of Prussia. (It is now Kaliningrad, on the Baltic coast in an exclave of Russia between Poland and Lithuania.) On Sunday afternoons, the good people of Königsberg would promenade about town, trying to walk across each of the seven bridges exactly once each. (See Figure 6.2.) No-one was ever able

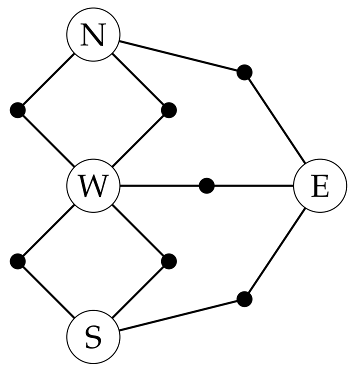


Figure 6.3: The graph of Königsberg.

to do it – eventually the mayor asked Leonard Euler why this seemed to be impossible. And so, in 1736, Euler explained why this was in fact impossible.

Let's form a bipartite graph as follows. On one side of the bipartition are the two banks of the river and the two islands. On the other side of the bipartition are the seven bridges. A bridge is joined to the two locations at either end of it. This gives the graph in Figure 6.3. The good people of Königsberg were trying to find a walk in this graph that traverses each edge exactly once.

Definition 6.16 (Euler Tour, Eulerian Graph)**.** An *Euler tour* in a graph is a walk which traverses each edge exactly once. A graph which has an Euler tour is an *Eulerian graph*.

The term "Euler tour" is conventional, but "Eulerian trail" would have been better. Our goal is to give a structural characterization of Eulerian graphs. Vertices of degree zero are irrelevant in the context of Euler tours, so we can safely assume that every vertex has degree at least one.

Theorem 6.17 (Euler Tours)**.** *Let G be a graph with no vertices of degree zero.*

(a) Then G is Eulerian if and only if it is connected and has at most two vertices of odd degree.

(b) If an Eulerian graph has no vertices of odd degree then every Euler tour is a closed trail.

(c) If an Eulerian graph has two vertices v, w of odd degree then every Euler tour is a trail with ends v and w.

Proof. To begin with, let $W = (v_0\, v_1 \dots v_k)$ be an Euler tour of G. Since G has no vertices of degree zero, every vertex is incident with at least one edge. Since every edge occurs in W, the vertex v_0 reaches every vertex of G. Therefore, G is connected. Now consider any vertex $w \in V$. Let $X = \{i : 1 \leq i \leq k-1 \text{ and } w = v_i\}$ and $Y = \{i : i \in \{0, k\} \text{ and } w = v_i\}$. Since every edge of G occurs exactly once in W, it follows that $\deg(w) = 2\,|X| + |Y|$. Therefore, the only vertices of G that could have odd degree are v_0 and v_k. If $v_0 \neq v_k$ then these two vertices both have odd degree, while if $v_0 = v_k$ then this vertex has even degree. This proves one direction of part (a): if G is Eulerian then G is connected and has at most two vertices of odd degree. This also implies parts (b) and (c).

For the converse direction of part (a), first consider the case that $G = (V, E)$ is connected and every vertex has even degree. We prove by induction on $|E|$ that G has a closed Euler tour. The basis of induction is the case $|E| = 0$, in which case $G = (\{v\}, \varnothing)$ is a vertex of degree zero, and (v) is a closed Euler tour of G. The induction step uses the case that G is a cycle – a 2-regular connected graph. In this case, starting at any vertex of G we can walk around the cycle in either direction to get a closed Euler tour of G. For the induction step, let G be a connected graph in which every vertex has even degree, which is not just a vertex of degree zero, and assume the result for such graphs with fewer edges than G.

Since G has no vertices of degree zero, every vertex has degree at least two. By Proposition 6.8, G contains a cycle C. Let $Z = (z_0\, z_1 \dots z_\ell)$ be an Euler tour of the cycle C. Let $H = G \setminus E(C)$. Then every vertex of H has even degree. But H might not be connected, and might even have some vertices of degree zero. Let $H_1, H_2, \dots, H_c$ be the connected components of H.

Every component H_i has at least one vertex in common with C. This is clear if H is connected, because H is a spanning subgraph of G. Otherwise, $V(H_i)$ is a proper subset of $V(G)$. Since G is connected, Theorem 6.15 implies that in G, $\partial V(H_i) \neq \varnothing$. Now, since H_i is a connected component of $H = G \setminus E(C)$, this implies that $E(C) \cap \partial V(H_i) \neq \varnothing$. Thus, H_i has at least

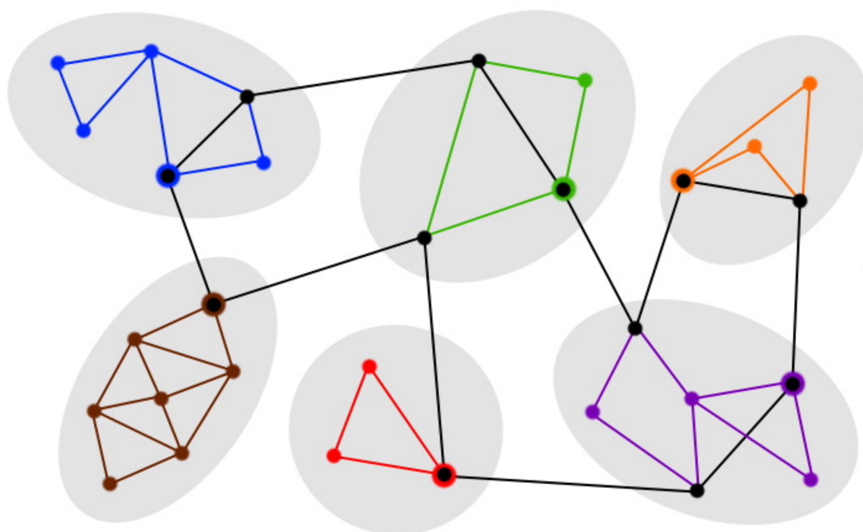


Figure 6.4: Proof of Theorem 6.17.

one vertex in common with C.

For each $1 \leq i \leq c$, let w_i be a vertex in $V(H_i) \cap V(C)$. By the induction hypothesis, each H_i has a closed Euler tour, which we may take to begin and end at w_i: denote this closed walk by W_i. Now construct a walk Q in G as follows. For each $1 \leq i \leq c$, the vertex w_i occurs exactly once on the closed walk Z (with the understanding that $z_0 = z_\ell$ is a single occurrence in cyclic order). Replace the occurrence of each vertex w_i in Z by the closed walk W_i to produce the closed walk Q. (See Figure 6.4 – the cycle C is in black and the various H_i and w_i are in different colours.) It is not hard to check that Q is a closed Euler tour of G, completing this part of the proof.

To finish the proof, what remains is the case that $G = (V, E)$ is connected and has exactly two vertices $v, w \in V$ of odd degree. Let z be a new vertex (not in V) and let $G' = (V \cup \{z\}, E \cup \{vz, wz\})$. Now G' is connected and every vertex has even degree. By the previous part of the proof, G' has a closed Euler tour which we may take to be of the form $(z\, v\, u_2\, u_3 \ldots u_{k-2}\, w\, z)$. Deleting the vertex z we obtain the (v, w)-trail $(v\, u_2\, u_3 \ldots u_{k-2}\, w)$, which is an Euler tour of G. This completes the proof. $\square$

Example 6.18. The graph of Königsberg has four vertices of odd degree, so it does not have an Euler tour.

6.4 Bridges / Cut-edges.

With the idea of connectedness in mind, certain edges of a graph play a special role. These are the edges which are essential to maintaining connectedness in the graph.

Definition 6.19 (Bridges)**.** An edge e of a graph G is a *bridge* if $G \smallsetminus e$ has strictly more components than G has: that is, $c(G \smallsetminus e) > c(G)$.

The words “cut-edge” or “coloop” or “isthmus” are synonyms for “bridge”.

The graphs pictured in Figures 5.7 to 5.10 give good illustrations of Theorem 6.20.

Theorem 6.20. *Let $G = (V, E)$ be a graph. An edge $e \in E$ is a bridge in G if and only if e is not contained in any cycles of G.*

Proof. Consider any edge $e \in E$, and let G_1 be the connected component of G that contains e. Then e is a bridge in G if and only if e is a bridge in G_1. Similarly, e is contained in a cycle of G if and only if e is contained in a cycle of G_1. Replacing G by G_1, if necessary, we can assume that G is connected.

First, assume that $e = xy$ is contained in a cycle C. Then $C \smallsetminus e$ supports a path $(x\,P\,y)$ in $G \smallsetminus e$ from x to y. To show that e is not a bridge we show that $G \smallsetminus e$ is connected. Let $v, w \in V$. Since G is connected, there is a (v, w)-walk in G. By Corollary 6.6, there is a (v, w)-path $Q = (v \ldots w)$ in G. If Q does not use the edge e then Q shows that v reaches w in $G \smallsetminus e$. If Q does use the edge e then this edge occurs exactly once (since paths are trails by Proposition 6.4). Exchanging the names of x and y if necessary, we may assume that Q has the form $Q = (v \ldots x\,y \ldots w)$. Replacing the step $(x\,y)$ with the path $(x\,P\,y)$ produces $(v \ldots x\,P\,y \ldots w)$, which is a (v, w)-walk in $G \smallsetminus e$. This shows that v reaches w in $G \smallsetminus e$ in this case as well. Hence, G is connected and e is not a bridge.

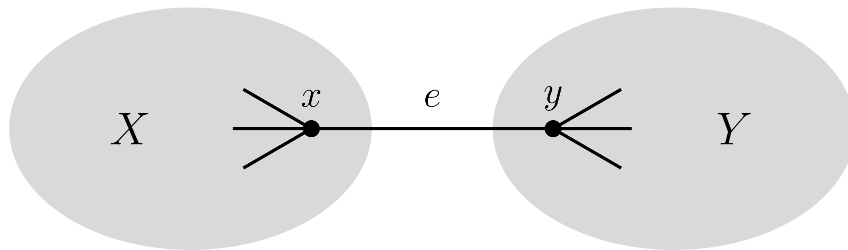


Figure 6.5: Schematic picture of a bridge.

Conversely, assume that $e = xy$ is not a bridge of G. Then $G \smallsetminus e$ is connected, so that $G \smallsetminus e$ contains an (x, y)-walk, and hence an (x, y)-path P (by Corollary 6.6). Now $C = (V(P), E(P) \cup \{e\})$ is a cycle in G that contains e, completing the proof. □

Figure 6.5 illustrates Proposition 6.21 and motivates the terminology.

Proposition 6.21. *Let $G = (V, E)$ be a connected graph, and let $e = xy \in E$ be a bridge. Then $G \smallsetminus e$ has exactly two connected components X and Y, with $x \in V(X)$ and $y \in V(Y)$.*

Proof. Let X be the connected component of $G \smallsetminus e$ that contains x, and let Y be the connected component of $G \smallsetminus e$ that contains y. We claim that $X \neq Y$ and that these are the only components of $G \smallsetminus e$.

Suppose that $X = Y$. Then there is an (x, y)-walk in $G \smallsetminus e$, and hence an (x, y)-path P in $G \smallsetminus e$ (by Corollary 6.6). As above, then $C = (V(P), E(P) \cup \{e\})$ is a cycle in G that contains e. By Theorem 6.20, this contradicts the hypothesis that e is a bridge in G. Therefore, $X \neq Y$.

To see that X and Y are the only components of $G \smallsetminus e$, consider any vertex $z \in V$. Since G is connected, there is a path P from x to z in G. If P does not use the edge e then z is in the component X of $G \smallsetminus e$. If P does use the edge $e = xy$ then this must be the first edge of P – for otherwise P would have to have x as a repeated vertex. That is, P has the form $P = (x\, y\, R\, z)$ and the path $(y\, R\, z)$ shows that z is in the component Y of $G \smallsetminus e$. □

Corollary 6.22. *If e is a bridge of a graph G then $c(G \smallsetminus e) = c(G) + 1$.*

Proof. Exercise 6.9. $\square$

6.5 Exercises.

Exercise 6.1. Let H be a subgraph of G. Prove that every connected component of H is a subgraph of some connected component of G.

Exercise 6.2. Fix an integer $k \geq 2$. Let G be a graph in which every vertex has degree at least k. Show that G contains (as a subgraph) a cycle with at least $k+1$ edges. (Hint: generalize Proposition 6.8.)

Exercise 6.3. Let G be a graph in which every vertex has degree at least 3. Show that G contains a cycle of even length.

Exercise 6.4. Let G be a graph that contains a closed walk of odd length. Show that G contains an odd cycle. (Hint: consider a closed walk of odd length that is as short as possible.)

Exercise 6.5. Let P and Q be two paths of maximum length in a connected graph G. Show that P and Q have a vertex in common. (Hint: use proof by contradiction.)

Exercise 6.6. Let $G = (V, E)$ be a graph. Let $C = (W, F)$ be a subgraph of G that is a cycle with an odd number of edges. Assume that C has the smallest number of edges among all odd cycles in G. Show that if v and w are vertices of C and vw is an edge of G, then vw is an edge of C. (That is, "a shortest odd cycle is an induced subgraph".)

Exercise 6.7. Let $C \neq C'$ be distinct cycles in a graph G, such that $E(C) \cap E(C') \neq \varnothing$. Show that there is a cycle of G with edge-set contained in the symmetric difference $E(C) \triangle E(C')$.

Exercise 6.8. Fix an integer $t \geq 1$.

(a) Show that if G is a graph with $2t + 1$ vertices, in which every vertex has degree at least t, then G is connected.
(b) Give an example of a graph G with $2t+1$ vertices, in which every vertex has degree at least $t - 1$, but such that G is not connected.

Exercise 6.9. Prove Corollary 6.22.

Exercise 6.10. Let $G = (V, E)$ be a graph, let $v \in V$ be a vertex in G of degree one, and let $e = vz \in E$ be the edge of G incident with v. Denote by $G \smallsetminus v$ the subgraph $(V \smallsetminus \{v\}, E \smallsetminus \{e\})$ of G.

(a) Show that G is connected if and only if $G \smallsetminus v$ is connected.
(b) Show that G contains a cycle if and only if $G \smallsetminus v$ contains a cycle.

Exercise 6.11. Prove the following about a graph G.

(a) If exactly two vertices u and v of G have odd degree, then G contains a path from u to v.
(b) If every vertex of G has even degree, then G does not contain a bridge.
(c) If every vertex of G has even degree, then the edges of G can be partitioned into edge-disjoint cycles.

Exercise 6.12. Let $G = (V, E)$ be a graph with no bridges. Consider the following relation $\approx$ defined on the set E: for $e, f \in E$, $e \approx f$ means that every cycle in G that contains e also contains f. Clearly the relation $\approx$ is reflexive (for all $e \in E$: $e \approx e$). This exercise completes the verification that $\approx$ is an equivalence relation on E. (The equivalence classes of this relation are called the *series classes* of E.)

(a) Show that $\approx$ is transitive: for all $e, f, g \in E$, if $e \approx f$ and $f \approx g$ then $e \approx g$.
(b) Show that $\approx$ is symmetric: for all $e, f \in E$, if $e \approx f$ then $f \approx e$.

Exercise 6.13. The *girth* of a graph G is the shortest length of a cycle in G. (If G has no cycles then the girth is infinite, by convention.) Fix integers $d \geq 2$ and $g \geq 3$. Let $G = (V, E)$ be a d-regular graph of girth g.

(a) Show that if $g = 4$ then $|V| \geq 2d$.
(b) For each $d \geq 2$, give an example of a d-regular graph of girth 4 with $2d$ vertices.
(c) Show that if $g = 5$ then $|V| \geq d^2 + 1$.
(d) Give examples of d-regular graphs of girth 5 and $d^2 + 1$ vertices for $d = 2$ and $d = 3$. (Another example is known to exist for $d = 7$, with 50 vertices. It is known that the only other value of d for which such a graph could exist is for $d = 57$, with $n = 3250$ vertices. But it is not known whether there is such a graph or not. These are called “Moore graphs”.)
(e) Show that if $d \geq 3$ and $g = 2k + 1 \geq 3$ is odd, then

$$|V| \geq \frac{d(d-1)^k - 2}{d - 2}.$$

(f) When $d \geq 3$ and $g = 2k \geq 4$ is even, give a lower bound on $|V|$ similar to the bound in part (e).
(g) Give an example meeting the bound in part (f) when $d = 3$ and $g = 6$.

Exercise 6.14. Recall the Odd graphs from Exercise 5.22. Fix $d \geq 3$, and let $X = \{1, 2, ..., 2d - 1\}$, so the vertices of $\mathbf{O}_d$ are the $(d-1)$-element subsets of X.

(a) Let $\beta : X \to X$ be any bijection. Define a function $f : V(\mathbf{O}_d) \to V(\mathbf{O}_d)$ by putting $f(S) = \{\beta(s) : s \in S\}$ for each vertex S of $\mathbf{O}_d$. Show that f is an automorphism of $\mathbf{O}_d$.
(b) For any two vertices v, w of $\mathbf{O}_d$, there is an automorphism $f : \mathbf{O}_d \to \mathbf{O}_d$ such that $f(v) = w$.

(c) For any two edges v_0v_1 and w_0w_1 of $\mathbf{O}_d$, there is an automorphism $f : \mathbf{O}_d \to \mathbf{O}_d$ such that $f(v_0) = w_0$ and $f(v_1) = w_1$.

(d) For any two paths $v_0v_1v_2$ and $w_0w_1w_2$ of length two in $\mathbf{O}_d$, there is an automorphism $f : \mathbf{O}_d \to \mathbf{O}_d$ such that $f(i) = w_i$ for $i \in \{0, 1, 2\}$.

(e) For any two paths $v_0v_1v_2v_3$ and $w_0w_1w_2w_3$ of length three in $\mathbf{O}_d$, there is an automorphism $f : \mathbf{O}_d \to \mathbf{O}_d$ such that $f(i) = w_i$ for $i \in \{0, 1, 2, 3\}$.

Chapter 7

Trees.

7.1 Trees and Minimally Connected Graphs.

When thinking about graphs as models of communication or transportation networks, it is natural to consider networks that are connected as efficiently as possible – that is, with no redundancy. This corresponds to a graph $G = (V, E)$ that is *minimally connected*: G is connected, but for every edge $e \in E$, $G \smallsetminus e$ is not connected. In other words, G is connected and every edge is a bridge.

Definition 7.1 (Trees, Forests, Leaves)**.**

1. A *tree* is a graph that is connected and contains no cycles.
2. A *forest* is a graph that contains no cycles. Every connected component of a forest is a tree.
3. A *leaf* is a vertex of degree one.

Proposition 7.2. *A graph is minimally connected if and only if it is a tree.*

Proof. If G is minimally connected then it is connected and since every edge of G is a bridge, Theorem 6.20 implies that G contains no cycles. Conversely, if G is a tree then G is connected. Since G contains no cycles, none of the edges of G are contained in cycles, so by Theorem 6.20 again, every edge of G is a bridge. □

Proposition 7.3. *A tree $T = (V, E)$ with at least two vertices has at least two leaves.*

Proof. Let $P = (v_0\, v_1 \ldots v_k)$ be a path in T of maximum length. Then P has at least one edge, since T is connected and has at least two vertices. Therefore, $v_0 \neq v_k$. If the degree of v_0 were bigger than one then v_0 would have a neighbour $w \neq v_1$. If w is not on the path P then $(w\, v_0\, v_1 \ldots v_k)$ is a path in T that is strictly longer than P, a contradiction. If $w = v_j$ is on the path P then $(v_0\, v_1 \ldots v_j\, v_0)$ is a cycle in T, contradicting the fact that T is a tree. Therefore, the degree of v_0 is equal to one. Similarly, the degree of v_k is also equal to one. □

Proposition 7.4. *A nonempty graph $G = (V, E)$ is a tree if and only if for any two vertices $v, w \in V$, there is exactly one path from v to w in G.*

Proof. First assume that G is a tree, and let $v, w \in V$. Since G is connected, there is a walk from v to w in G, so there is at least one path from v to w in G (by Corollary 6.6). If there are two distinct paths from v to w in G then G contains a cycle (by Proposition 6.7). Since G has no cycles, there is at most one path from v to w.

Conversely, if G is nonempty but not a tree then either G has more than one component or G contains a cycle. If G is not connected then let $v, w \in V$ be in different connected components of G. Then there is no path in G from v to w. If G contains a cycle $C = (v_0\, v_1\, v_2 \ldots v_k\, v_0)$ then $(v_0\, v_1\, v_2 \ldots v_k)$ and $(v_0\, v_k)$ are two different paths from v_0 to v_k in G. □

Theorem 7.5. *Let $G = (V, E)$ be a graph with $|V| = n$ vertices, $|E| = m$ edges, and $c(G) = c$ components. Then $m \geq n - c$, and equality holds if and only if G is a forest.*

Proof. We prove the result by induction on $m = |E|$. For the basis $m = 0$ of induction, $G = (V, \varnothing)$ is an edgeless graph with $|V| = n$ vertices and $c(G) = |V| = n$ components, and it is a forest since it contains no cycles.

For the induction step, assume that G has at least one edge, and that the result holds for all graphs with fewer edges than G. Let $e \in E$ be any edge of G, and consider the spanning subgraph $G' = G \smallsetminus e$ with $n' = n$ vertices, $m' = m-1$ edges, and $c' = c(G \smallsetminus e)$ connected components. By the induction hypothesis, $m' \geq n' - c'$ and equality holds if and only if G' is a forest.

First, assume that e is a bridge in G, so that $c' = c+1$ by Corollary 6.22. It follows that $m = m'+1 \geq (n'-c')+1 = n-(c'-1) = n-c$, proving part of the induction step. Also, $m = n - c$ if and only if $m' = n' - c'$. By induction, this occurs if and only if G' is a forest. We claim that since e is a bridge, G is a forest if and only if $G \smallsetminus e$ is a forest. Proof of this claim is left as Exercise 7.2, and completes this part of the induction step.

Second, assume that e is not a bridge in G, so that $c' = c$. It follows that $m = m'+1 \geq (n'-c')+1 = n-c+1 > n-c$. The claimed inequality holds, but not with equality. Since e is not a bridge it is contained in a cycle (by Theorem 6.20), and so G is not a forest. This completes the induction step and the proof. □

The special case of Theorem 7.5 for connected graphs ($c = 1$) is particularly important.

Corollary 7.6. *Let $G = (V, E)$ be a connected graph with $|V| = n$ vertices and $|E| = m$ edges. Then $m \geq n - 1$, and equality holds if and only if G is a tree.*

Example 7.7 (Numerology of Trees)**.** Let $T = (V, E)$ be a tree with n_d vertices of degree d, for all $d \in \mathbb{N}$. Then

$$n = |V| = n_0 + n_1 + n_2 + n_3 + \cdots$$

and, by the Handshake Lemma,

$$2m = 2\,|E| = n_1 + 2n_2 + 3n_3 + \cdots .$$

Since $|E| = |V| - 1$ by Corollary 7.6 we have

$$2(n_0 + n_1 + n_2 + n_3 + \cdots) = 2\,|V| = 2 + 2\,|E| = 2 + n_1 + 2n_2 + 3n_3 + \cdots .$$

This yields
$$2n_0 + n_1 = 2 + n_3 + 2n_4 + 3n_5 + \cdots$$
If $n = 1$ then $n_0 = 1$ and $n_d = 0$ for all $d \geq 1$. If $n \geq 2$ then $n_0 = 0$, and we deduce that $n_1 \geq 2$. This gives a second (more quantitative) proof of Proposition 7.3.

Theorem 7.8 (Two-out-of-Three Theorem). *Let $G = (V, E)$ be a graph with $|V| = n$ vertices and $|E| = m$ edges. Consider the following three conditions:*
(i) *G is connected;*
(ii) *G has no cycles;*
(iii) $m = n - 1$.
Any two of these conditions together imply that all three hold.

Proof. If (i) and (ii) hold then G is a tree. Corollary 7.6 implies that (iii) holds.

If (i) and (iii) hold then the inequality of Corollary 7.6 holds with equality, so that G is a tree and hence has no cycles.

If (ii) and (iii) hold then let $G_1, G_2, ..., G_c$ be the connected components of G. Let G_i have n_i vertices and m_i edges. Each G_i is connected and by (ii) has no cycles, so is a tree, so that $m_i = n_i - 1$ by Corollary 7.6. Now (iii) implies that

$$\begin{aligned} 1 = n - m &= (n_1 + n_2 + \cdots + n_c) - (m_1 + m_2 + \cdots + m_c) \\ &= (n_1 - m_1) + (n_2 - m_2) + \cdots + (n_c - m_c) = c. \end{aligned}$$

Since $c = 1$, G is connected, and so (i) holds. This completes the proof. □

7.2 Spanning Trees and Connectedness.

A *spanning tree* of a graph is just what it sounds like: it is a spanning subgraph that is a tree.

Theorem 7.9. *A graph is connected if and only if it has a spanning tree.*

Proof. First, assume that T is a spanning tree of G. Since T is connected, it is nonempty, so G is nonempty. Let $v, w \in V(G)$. Since T is spanning and connected, v reaches w in T. Since T is a subgraph of G, this implies that v reaches w in G. Therefore, G is connected.

Conversely, assume that $G = (V, E)$ is connected, with $|V| = n$ vertices and $|E| = m$ edges. By Corollary 7.6, $m \geq n - 1$, with equality if and only if G is a tree. We proceed by induction on m. The basis of induction is $m = n - 1$. In this case, G is a tree, and hence is a spanning tree of itself. Assume that the result holds for all connected graphs G' with $V(G') = V(G)$ and $|E(G')| < |E(G)|$. For the induction step, G is connected and $m \geq n > n - 1$. By Corollary 7.6 again, G is not a tree. Thus, G contains a cycle C. Let $e \in E$ be an edge of C and let $G' = G \smallsetminus e$. By Theorem 6.20, the edge e is not a bridge of G, so that G' is connected. By the induction hypothesis, G' contains a spanning tree T. Since G' is a spanning subgraph of G, it follows that T is also a spanning tree of G. $\square$

In Proposition 5.25(d) we saw that if G contains an odd cycle then G is not bipartite. Using spanning trees we can give an efficient proof of the converse implication.

Theorem 7.10. *A graph is bipartite if and only if it does not contain any odd cycles.*

Proof. First, notice that a graph is bipartite if and only if every one of its connected components is bipartite. Also, a graph contains an odd cycle if and only if at least one of its connected components contains an odd cycle. These observations allow us to reduce the proof of this theorem to the connected case. That is, if the statement holds for all connected graphs then it holds for all graphs. Thus, we may assume that $G = (V, E)$ is a connected graph.

If G contains an odd cycle then G is not bipartite, by Proposition 5.25(d). Conversely, assume that G is not bipartite. Since G is connected, Theorem 7.9 implies that G has a spanning tree T. An easy induction shows that trees are bipartite (this is left as Exercise 7.3). Let (A, B) be a bipartition of T. Since G is not bipartite, this is not a bipartition of G. Thus, there is an edge $vw \in E$ with both ends in A or both ends in B. By symmetry, we may assume that $v, w \in A$. By Proposition 7.4 there is exactly one (v, w)-path P in T. Since both ends of P are in the set A of the bipartition (A, B) of T, the path P has an even number of steps. Now $C = (V(P), E(P) \cup \{vw\})$ is a cycle in G with an odd number of edges, completing the proof. □

The proof of Theorem 7.9 suggests the following strategy for finding a spanning tree of a connected graph. Given a connected graph G, if G contains a cycle then delete an edge of that cycle. Repeat this until what is left has no cycles. Then what is left is a spanning tree of G. This is difficult to turn into an algorithm because of the phrase "if G contains a cycle". We would need a subroutine that takes a connected graph G as input, and produces as output either a cycle in G or a certificate that G contains no cycles. To do this, it seems to be easier just to find a spanning tree first.

Algorithm 7.11 (Constructing a spanning forest).

```
Input: a graph G = (V, E);
let F := ∅; let n := 0;
for v ∈ V do { let n := n + 1; let g(v) := n; };
for e = vw ∈ E do
{  if g(v) ≠ g(w) then
   {  let F := F ∪ {e};
      let a := g(v); let b := g(w);
      for z ∈ V do { if g(z) = b then let g(z) := a; };
   };
};
Output: (V, F) and g : V → {1, 2, ..., n}.
```

Informally, Algorithm 7.11 does the following. The loop over vertices $v \in V$ defines an arbitrary bijection $g : V \to \{1, 2, ..., n\}$. The values of g will be used to label the components of the output spanning forest. The loop over edges $vw \in E$ tests if an edge has the same label at both ends: $g(v) = g(w)$. If so, then that edge is skipped. But if $g(v) \neq g(w)$ then the edge is included in the output, and all of the vertices labelled $g(w)$ are re-labelled to have the same label as $v \in V$. After all edges have been examined, the algorithm stops and produces its output.

Theorem 7.12. *Let $G = (V, E)$ be a graph. Apply Algorithm 7.11 to the input $G = (V, E)$, and let (V, F) and $g : V \to \{1, 2, ..., n\}$ be the corresponding output.*

(a) Then (V, F) is a spanning forest of G.
(b) Vertices $v, w \in V$ are in the same component of (V, F) if and only if $g(v) = g(w)$.
(c) Each component of (V, F) is a spanning tree of a component of G.

Proof. To prove part (a) we only need to show that (V, F) does not contain a cycle. We prove this together with part (b) by showing that these statements hold after every passage through the loop over $e = vw \in E$. Before this loop starts we have $F = \varnothing$, and both statements (a) and (b) hold.

Now consider one iteration of the loop, and the edge $e = vw \in E$ which is being examined. At this stage the subgraph (V, F) and function g satisfy statements (a) and (b), by induction on the number of passages through the loop. There are two cases. If $g(v) = g(w)$ then the set F and function g do not change, and so statements (a) and (b) continue to hold. If $g(v) \neq g(w)$ then let $F' = F \cup \{vw\}$ be the updated set F, and let $g' : V \to \{1, 2, ..., n\}$ be the updated function. Since (b) holds for (V, F), the vertices v and w are in different components of (V, F). But v and w are in the same component of (V, F'). Therefore, $e = vw$ is a bridge of (V, F'). By Exercise 7.2, since (V, F) is a forest it follows that (V, F') is a forest. Because of the way g' is updated, and becuase (b) holds for (V, F) and g, it follows that for all $z \in V$, $g'(z) = g'(v)$ if and only if z is in the component of (V, F') containing v. From this the statement (b) follows for the subgraph (V, F') and function g'. This completes the induction step, and the proof of parts (a) and (b).

For part (c), since (V, F) is a subgraph of G, every component of (V, F) is contained in a component of G. To prove (c), it suffices to show that if x reaches y in G then x reaches y in (V, F). To do this the following observation is useful: if at any stage of the algorithm we have $g(x) = g(y)$, then this continues to hold at all later stages. (This can be proved by induction on the number of iterations of the loop over the edges.) The contrapositive form is even more useful: if at any stage of the algorithm we have $g(x) \neq g(y)$, then $g(x) \neq g(y)$ also held at all earlier stages.

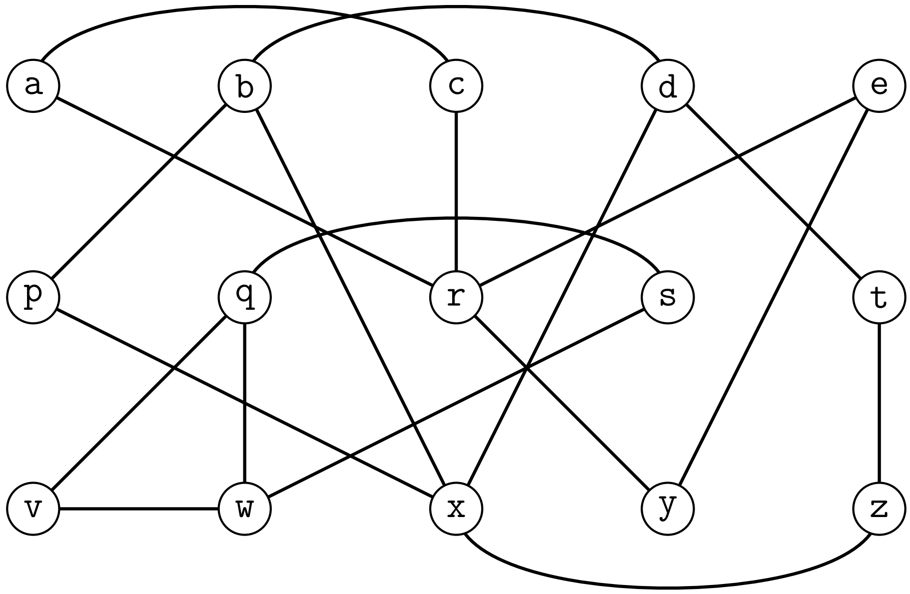


Figure 7.1: Graph for Examples 7.13 and 7.16.

Now suppose that $x, y \in V$ are such that x reaches y in G, but that x and y are in different components of the output (V, F). Since we have proved (b), we know that $g(x) \neq g(y)$. Let W be an (x, y)-walk in G. Then there is an edge $e = vw$ on W for which $g(v) \neq g(w)$. Since (a) and (b) hold, it follows that e is not an edge in F. Now consider the iteration of the loop at which the edge $e = vw$ is considered. By the observation in the previous paragraph, at this stage we also have $g(v) \neq g(w)$. But then the algorithm would have included e in the set F. This contradiction shows that if x reaches y in G, then x reaches y in (V, F), completing the proof. $\square$

Example 7.13. Table 7.1 shows Algorithm 7.11 applied to the graph pictured in Figure 7.1. The columns are indexed by the vertices of the graph. The first row indicates the initial (arbitrary) values of the function g. In the first column, the remaining rows are indexed by the edges, in the order they are considered by the algorithm. In the row corresponding to an edge, the first-named vertex is marked with an asterisk (*). If the labels of the two ends are equal, this is marked with an equal sign ($=$) at

E	a	b	c	d	e	p	q	r	s	t	v	w	x	y	z	F
$g(\cdot)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
ac	$*$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	ac
ws	.	.	.	.	.	.	.	.	12	.	.	$*$	.	.	.	ws
dt	.	.	.	$*$	.	.	.	.	.	4	.	.	.	.	.	dt
cr	.	.	$*$	.	.	.	.	1	.	.	.	.	.	.	.	cr
vw	.	.	.	.	.	.	.	.	11	.	$*$	11	.	.	.	vw
pb	.	6	.	.	.	$*$	.	.	.	.	.	.	.	.	.	pb
ye	.	.	.	.	14	.	.	.	.	.	.	.	.	$*$	.	ye
bx	.	$*$	.	.	.	.	.	.	.	.	.	.	6	.	.	bx
xz	.	.	.	.	.	.	.	.	.	.	.	.	$*$	.	6	xz
qw	.	.	.	.	.	.	$*$	.	7	.	7	7	.	.	.	qw
ar	$*$	.	.	.	.	.	.	$=$	.	.	.	.	.	.	.	
er	14	.	14	.	$*$	.	.	14	.	.	.	.	.	.	.	er
vq	.	.	.	.	.	.	$=$	.	.	.	$*$	.	.	.	.	
bd	.	$*$	.	6	.	.	.	.	.	6	.	.	.	.	.	bd
qs	.	.	.	.	.	.	$*$	.	$=$	.	.	.	.	.	.	
ry	.	.	.	.	.	.	.	$*$	.	.	.	.	.	$=$	.	
tz	.	.	.	.	.	.	.	.	.	$*$	.	.	.	.	$=$	
px	.	.	.	.	.	$*$	.	.	.	.	.	.	$=$	.	.	
$g(\cdot)$	14	6	14	6	14	6	7	14	7	6	7	7	6	14	6	

Table 7.1: Algorithm 7.11 in action.

the other end. Otherwise, the numbers in that row indicate the process of re-labelling the vertices, and the last column contains the name of the edge, which is included in the output. The subgraph (V, F) output from this example is shown in Figure 7.2.

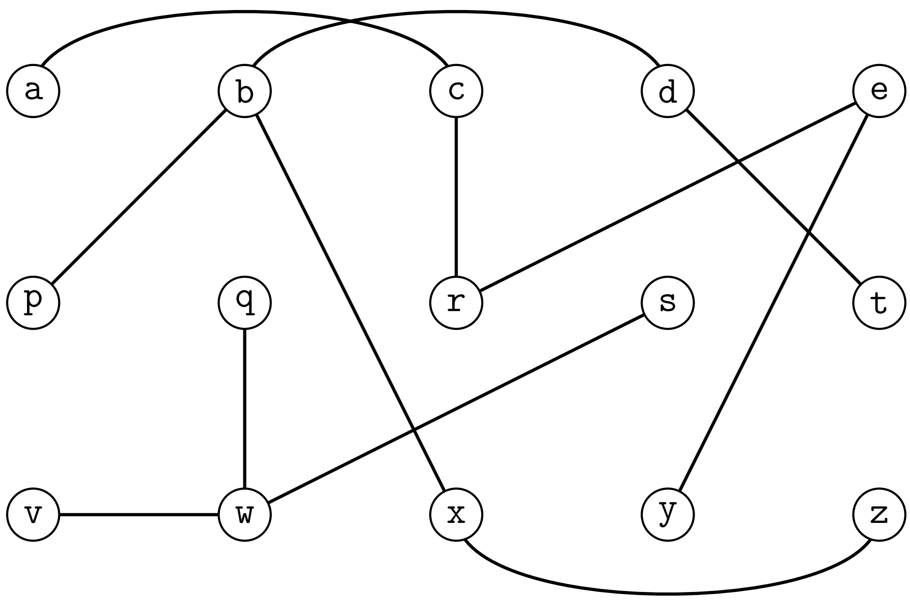


Figure 7.2: The subgraph output for Example 7.13.

7.3 Search Trees.

In this section we give an algorithm that produces a spanning tree T of a connected graph G, along with some extra information that helps us navigate within the graph. This extra information is a "root vertex" v_*, a "parent function" pr, and a "level function" ℓ. These are explained below.

Recall the concept of the boundary of a subset $S \subseteq V$ in a graph $G = (V, E)$: this is the set $\partial S = \{e \in E : \ |e \cap S| = 1\}$ of edges with exactly one end in S. We will need to calculate ∂S from S in our algorithm. Clearly this can be done using a loop over the set of all edges of G, and we will use this as a subroutine without going into details. (This whole algorithm can be implemented more efficiently than we do here, using clever data structures, but we are not concerned with issues of computational complexity.)

Algorithm 7.14 (Constructing a search tree)**.**
Input: a graph $G = (V, E)$ and a vertex $v_* \in V$;
let $W := \{v_*\}$; let $F := \varnothing$;

let $\mathrm{pr}(v_*) := \texttt{null}$; let $\ell(v_*) := 0$;
let $\Delta := \partial W$;
while ($\Delta \neq \varnothing$) do
$\{$ pick any $e = xy \in \Delta$ with $x \in W$ and $y \notin W$;
  let $W := W \cup \{y\}$; let $F := F \cup \{e\}$;
  let $\mathrm{pr}(y) := x$; let $\ell(y) := 1 + \ell(x)$;
  let $\Delta := \partial W$;
$\}$;
Output: $T = (W, F)$ and $\mathrm{pr} : W \to W \cup \{\texttt{null}\}$ and $\ell : W \to \mathbb{N}$.

This is a non-deterministic algorithm because of the phrase "pick any $e = xy \in \Delta$". By restricting the choice of edge from Δ in various ways one can specialize to particular kinds of search trees which have extra useful properties. This is discussed in Chapter **??**.

Theorem 7.15 (Search Trees)**.** *Let $G = (V, E)$ be a graph, and let $v_* \in V$. Apply Algorithm 7.14 to the input G and v_*, and let $T = (W, F)$, $\mathrm{pr} : W \to W \cup \{\texttt{null}\}$, and $\ell : W \to \mathbb{N}$ be the corresponding output.*

(a) Then $T = (W, F)$ is a spanning tree of the connected component of G that contains v_*.
(b) For any $w \in W$, the unique path P in T from w to v_* is obtained by following the steps $v \to \mathrm{pr}(v)$ until $\mathrm{pr}(v) = \texttt{null}$.
(c) For any $w \in W$, the length of the path P from w to v_* in T is $\ell(w)$.

Proof. We prove that T is a tree and that parts (b) and (c) hold by induction on the number of iterations through the "while ($\Delta \neq \varnothing$)" loop. At the end we finish the proof that (a) holds.

Before the loop begins, $T = (\{v_*\}, \varnothing)$ is a tree, the functions pr and ℓ are defined on the set W, and parts (b) and (c) hold. Now consider an iteration of the loop in which the edge $e = xy \in \Delta$ is considered. Let $T = (W, F)$ and pr and ℓ be the data before the edge e is added, and let $T' = (W', F')$ and pr' and ℓ' be the data after the edge e is added. By induction, $T = T' \smallsetminus y$ is a tree. By construction, y is a vertex of degree one in T'. Exercise 6.10 (or the Two-out-of-Three Theorem 7.8) now implies that T' is a tree. Parts (b) and (c) only need to be checked for the new vertex y. By induction, the unique path in T from x to v_* is obtained by following the steps $v \to \mathrm{pr}(v)$ until $\mathrm{pr}(v) = \texttt{null}$, and the length of this path is $\ell(x)$. It follows that the unique

F	W	$\ell(\cdot)$	∂W
$\varnothing$	`x`	0	`xp`, `xb`, `xd`, `xz`
`xb`	`b`	1	`xp`, `xd`, `xz`, `bp`, `bd`
`xz`	`z`	1	`xp`, `xd`, `bp`, `bd`, `zt`
`bp`	`p`	2	`xd`, `bd`, `zt`
`bd`	`d`	2	`zt`, `dt`
`zt`	`t`	2	$\varnothing$

Table 7.2: Algorithm 7.14 in action.

path in T' from y to v_* is obtained by following the steps $v \to \mathrm{pr}'(v)$ until $\mathrm{pr}'(v) = \texttt{null}$, and the length of this path is $\ell'(y) = 1 + \ell(x)$.

To see that T is a spanning tree of the component of G containing v_*, suppose that v_* reaches $z \in V$ in G but that $z \notin W$ when the algorithm terminates. Let Z be a (v_*, z)-walk in G. Since $v_* \in W$ and $z \notin W$, there is a step xy of Z with $x \in W$ and $y \notin W$. But this edge $e = xy$ is in the boundary of W. Now, since $\partial W \neq \varnothing$, the algorithm has not terminated yet. This contradiction shows that if v_* reaches $z \in V$ in G, then $z \in W$. Thus, T is a spanning tree of the component of G containing v_*. □

Example 7.16. Table 7.2 shows Algorithm 7.14 applied to the graph pictured in Figure 7.1 with root vertex `x`. The first row indicates the data initialization stage. The first column indicates the edges included in the output set F, in the order they are chosen by the algorithm. The second column indicates the new vertex that is adjoined to the set W at each step, and the third column is the level of the new vertex. The fourth column is the boundary ∂W of the updated set W. The output from this example is shown in Figure 7.3.

7.4 Exercises.

Exercise 7.1. Draw pictures of all trees with six or fewer vertices (up to isomorphism). (Be brave and do seven!)

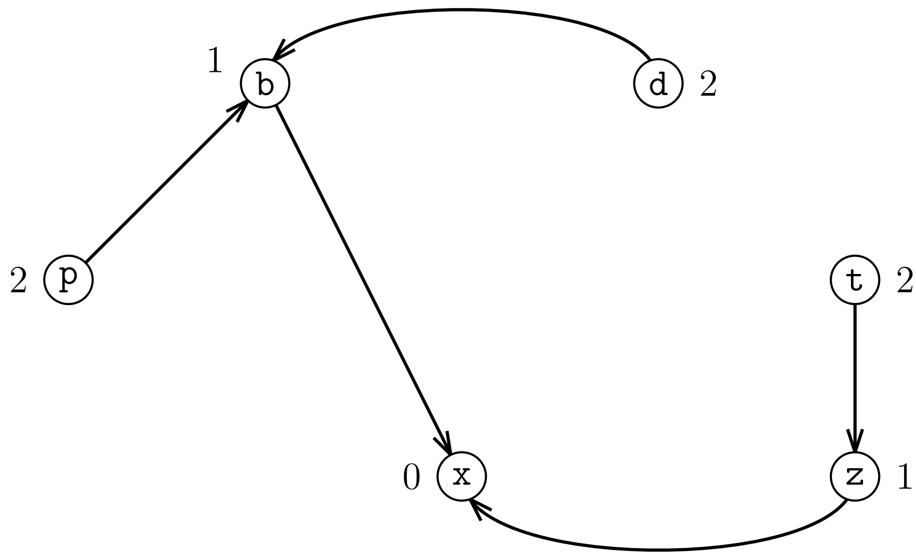


Figure 7.3: The output for Example 7.16.

Exercise 7.2. Let $e \in E$ be a bridge in a graph $H = (V, E)$. Show that H is a forest if and only if $H \smallsetminus e$ is a forest.

Exercise 7.3. Prove that trees are bipartite. (Hint: go by induction on the number of vertices, by deleting a leaf.)

Exercise 7.4. Let $G = (V, E)$ be a graph.

(a) Show that if $|E| = |V| - 1$ then G has a vertex of degree at most one.

(b) Use part (a) and Exercise 6.10 to give a proof of the Two-out-of-Three Theorem 7.8 that does not use Theorem 7.5 or its consequences. (You can use Proposition 7.3.)

Exercise 7.5. Fix an integer $d \geq 3$, and let T be a tree with n_1 vertices of degree one, n_d vertices of degree d, and no other vertices. Determine $|V(T)| = n_1 + n_d$ as a function of n_1. For which values of n_1 does such a tree exist?

Exercise 7.6. Let T be a tree with $n \geq 2$ vertices and no vertices of degree 2. What is the minimum possible number of leaves that such a tree can have, as a function of n? Which trees attain this minimum bound?

Exercise 7.7. Let $\mathbf{d} = (d_1\ d_2\ \dots\ d_n)$ be a weakly decreasing sequence of $n \geq 2$ positive integers. Show that there is a tree with degree sequence $\mathbf{d}$ if and only if $d_1 + d_2 + \cdots + d_n = 2n - 2$.

Exercise 7.8. Let $G = (V, E)$ be a connected graph for which $|V| = |E|$. Prove that G contains exactly one cycle.

Exercise 7.9. Let e be an edge in a connected graph G. Prove that e is a bridge in G if and only if e is in every spanning tree of G.

Exercise 7.10. Let $G = (V, E)$ be a connected graph, and let T_1 and T_2 be two spanning trees of G.

(a) Let $e \in E(T_1) \smallsetminus E(T_2)$. Prove that there is an edge $f \in E(T_2) \smallsetminus E(T_1)$ such that $(T_1 \smallsetminus \{e\}) \cup \{f\}$ is also a spanning tree of G.

(b) Let $e \in E(T_1) \smallsetminus E(T_2)$. Prove that there is an edge $f \in E(T_2) \smallsetminus E(T_1)$ such that $(T_2 \cup \{e\}) \smallsetminus \{f\}$ is also a spanning tree of G.

Exercise 7.11. Describe an algorithm that takes as input a graph $G = (V, E)$ and produces as output a subset $R \subseteq V$ and a collection of subsets $\{U_r : r \in R\}$ with the following properties.

- The set R contains exactly one vertex from each component of G.
- For each $r \in R$, U_r is the set of vertices in the same component as r.

Exercise 7.12. In Algorithm 7.14, the line "let $\Delta := \partial W$;" can be replaced by "let $\Delta := \Delta \oplus \partial\{y\}$;". (Here, $\oplus$ denotes the symmetric difference of sets.) Explain why.

Chapter 8

Planar Graphs.

We consider the question: which graphs can be drawn in the plane without crossing edges? This is not only interesting in its own right, but also has practical implications. For instance, in the manufacture of very large-scale integrated circuits, electrical conductors are printed onto a polymer substrate. These conductors are the edges of a graph, and to avoid short circuits these edges must not cross one another. Another example is a map in which cities, towns, and crossroads are represented by vertices, and highways are represented by edges

Several results of this chapter are stated most naturally in terms of (undirected) multigraphs, and so we adopt this point of view from the beginning. For this chapter, the word "graph" will tacitly mean an undirected multigraph. Where necessary, we will be clear about restricting to simple graphs.

8.1 Plane Embeddings of Graphs.

What do we mean by a drawing of a graph? A *simple (parameterized plane) curve* is an injective continuous function $\gamma : [0, 1] \to \mathbb{R}^2$. We will conflate a simple curve with its image in $\mathbb{R}^2$, since any particular parameterization will turn out to be irrelevant. The endpoints of γ are $\gamma(0)$ and $\gamma(1)$, and $\gamma(1/2)$ is one of infinitely many midpoints. A *simple closed curve* is a continuous function $\gamma : [0, 1] \to \mathbb{R}^2$ that is almost injective, the only exception being that $\gamma(0) = \gamma(1)$.

Definition 8.1 (Plane Embedding, Planar Graph)**.** Let $G = (V, E, B)$ be an undirected multigraph. A *plane embedding* of G is a pair of sets $(\mathcal{P}, \Gamma)$ such that:

1. $\mathcal{P} = \{p_v : v \in V\}$ is a set of distinct points in $\mathbb{R}^2$ indexed by V.
2. $\Gamma = \{\gamma_e : e \in E\}$ is a set of distinct simple curves in $\mathbb{R}^2$ indexed by E.
3. For $e \in E$, the endpoints $\gamma_e(0)$ and $\gamma_e(1)$ of γ_e are the points p_v for which $B(v, e) > 0$. Moreover, $\gamma_e(0) = \gamma_e(1)$ if and only if e is a loop at v.
4. For $v \in V$ and $e \in E$, if $p_v \in \gamma_e$ then $B(v, e) > 0$ and $p_v \in \{\gamma_e(0), \gamma_e(1)\}$.
5. For $e, f \in E$ with $e \neq f$, if $0 \leq s, t \leq 1$ and $\gamma_e(s) = \gamma_f(t)$ then $s, t \in \{0, 1\}$.

A graph is *planar* if it has a plane embedding, and otherwise it is *non-planar*.

Condition 3. says that the curves representing edges have the right endpoints. Condition 4. says that edges don't go through vertices except at their ends. Condition 5. says that distinct edges cannot intersect except at their endpoints.

Here is a link to an amusing puzzle based on plane embeddings of graphs.

Example 8.2.

- The complete graphs K_1, K_2, K_3, K_4 are planar, but K_5 seems to be non-planar. (This is proved in Example 8.20.)
- The complete bipartite graphs $K_{1,r}$ and $K_{2,r}$ are planar, but $K_{3,3}$ seems to be non-planar. (This is proved in Example 8.22.)

To avoid topological complications, we usually restrict the simple curves in a plane embedding of a graph. For instance, there are injective functions $\gamma : [0, 1] \to \mathbb{R}^2$ which are continuous but nowhere differentiable. The limit of the process shown in Figure 8.2 is such an example, known as the *Koch snowflake curve*. To eliminate such pathologies, we require that an edge of a plane embedding is represented by a *tame curve*. Various conventions are used: if γ is continuously differentiable to all orders, or if γ is a concatenation

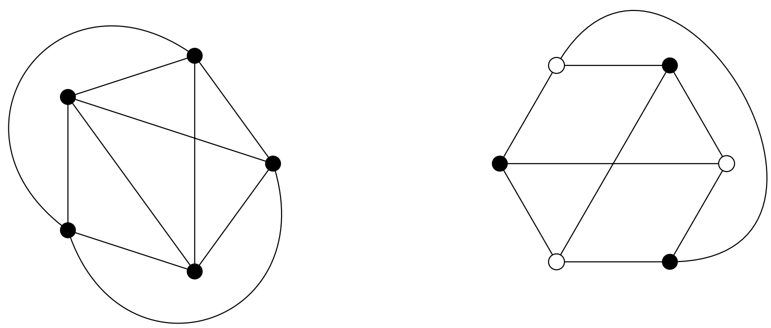


Figure 8.1: K_5 and $K_{3,3}$ seem to be non-planar.

of finitely many straight line segments, then γ is a tame curve. This will suffice for our purposes.

Theorem 8.3 (Jordan Curve Theorem)**.** *Let γ be a simple closed curve in $\mathbb{R}^2$. Then the complementary subset $\mathbb{R}^2 \smallsetminus \gamma$ has exactly two connected components, the interior of γ and the exterior of γ.*

For tame curves Theorem 8.3 is evident, and it was not until the early 1800s that people realized that this statement requires proof. For non-tame ("wild")

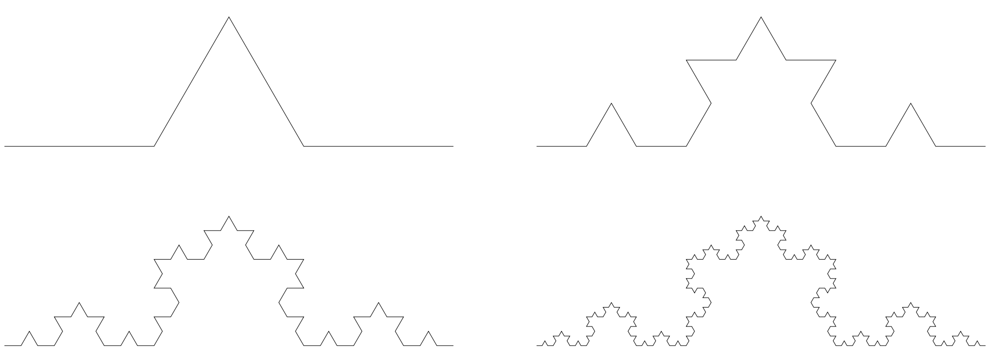


Figure 8.2: The Koch snowflake curve.

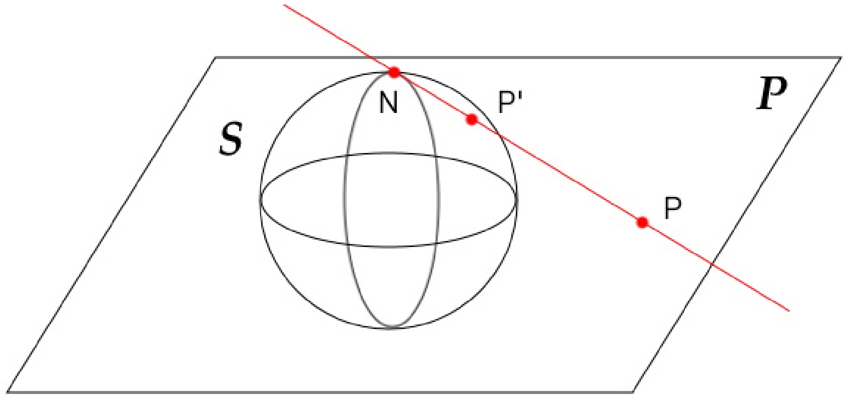


Figure 8.3: Stereographic projection.

curves, the proof is surprisingly tricky. Details can be found in any introductory book on point-set topology. We say that a simple closed curve *separates* the points in its interior from the points in its exterior.

8.1.1 Stereographic projection.

A plane embedding of a graph can be transferred to an embedding of the graph on the surface of a sphere, as follows. Informally, draw the graph on a piece of paper and then put the piece of paper on the floor. The graph is now drawn on the surface of the Earth, which is a sphere. The formal description of this process is known as *stereographic projection*.

Identify $\mathbb{R}^2$ with the plane $\mathbb{P} = \{(x, y, 0) : (x, y) \in \mathbb{R}^2\}$ in three-space in the obvious way. Let $\mathbb{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$, and let $\mathsf{N} = (0, 0, 1)$ be the "north pole" of $\mathbb{S}$. For any point P in the plane $\mathbb{P}$, there is a unique line $\overline{\mathsf{PN}}$ in $\mathbb{R}^3$ containing the points N and P. This line intersects the sphere $\mathbb{S}$ in two distinct points N and P'. This construction $\mathsf{P} \mapsto \mathsf{P}'$ defines a continuous bijection from the plane $\mathbb{P} \simeq \mathbb{R}^2$ to the punctured sphere $\mathbb{S} \smallsetminus \mathsf{N}$. (See Figure 8.3.) The exterior of the "equator" $\mathbb{S} \cap \mathbb{P}$ in $\mathbb{P}$ is mapped to the "northern hemisphere" of $\mathbb{S}$, and the interior of $\mathbb{S} \cap \mathbb{P}$ is mapped to the "southern hemisphere". In this way, any simple curve can be transferred between $\mathbb{R}^2$ and $\mathbb{S} \smallsetminus \mathsf{N}$, and so any plane embedding of a graph can be transferred similarly.

Stereographic projection is useful because one can take a plane embedding of a graph, project it onto the sphere, then rotate the sphere in three-space so that any point in the complement of the embedding is at the north pole, and then project the embedding from the sphere back onto the plane. This is an extra symmetry of plane embeddings that is not apparent at first glance.

8.2 Kuratowski's Theorem.

Which graphs can be drawn in the plane? In Example 8.2 we saw that K_5 and $K_{3,3}$ seem to be non-planar. This is the starting point for a complete characterization of planar graphs.

Lemma 8.4. *If G is planar then every subgraph of G is planar.*

Proof. If $(\mathcal{P}, \Gamma)$ is a plane embedding of G then

$$(\{p_w : \ w \in V(H)\}, \{\gamma_e : \ e \in E(H)\})$$

is a plane embedding of the subgraph H. □

Lemma 8.5. *A multigraph G is planar if and only if its simplification* $\mathrm{si}(G)$ *is planar.*

Proof. Since $\mathrm{si}(G)$ can be regarded as a subgraph of G, Lemma 8.4 implies one direction. The converse implication is Exercise 8.1(a). □

Definition 8.6 (Subdivision)**.** Let $G = (V, E, B)$ be a multigraph, and let $vew \in E$. Let z be a new vertex not in V, and let e' and e'' be new edges not in G. The *subdivision of e in G* is the multigraph $G \bullet e$ with vertices $V(G \bullet e) = V(G) \cup \{z\}$, edges $E(G \bullet e) = (E(G) \smallsetminus \{e\}) \cup \{e', e''\}$, and incidences $ve'z$ and $we''z$ for the new edges.

Informally, subdivision inserts a new vertex in the middle of the edge e. For example, the graph of Figure 5.15 can be obtained by repeated subdivision of edges starting from the graph $C_4 \,\square\, P_3$.

Lemma 8.7. *Let G be a multigraph and $e \in E(G)$. Then G is planar if and only if $G \bullet e$ is planar.*

Proof. First, assume that $(\mathcal{P}, \Gamma)$ is a plane embedding of G, and let $vew \in E$ with $p_v = \gamma_e(0)$ and $p_w = \gamma_e(1)$. Let $p_z = \gamma_e(1/2)$, and define two simple curves by $\gamma_{e'}(t) = \gamma_e(t/2)$ and $\gamma_{e''}(t) = \gamma_e(1 - t/2)$ for $t \in [0, 1]$. Together with the points and curves from the embedding of $G \setminus e$, this gives a plane embedding of $G \bullet e$. The converse implication is Exercise 8.1(b). □

Theorem 8.8 (Kuratowksi, 1930)**.** *A multigraph is planar if and only if it does not contain a (repeated) subdivision of K_5 or $K_{3,3}$ as a subgraph.*

Proof of the easy implication. By Examples 8.20 and 8.22 (below), the graphs K_5 and $K_{3,3}$ are non-planar. By Lemmas 8.4 and 8.7, if G contains a (repeated) subdivision of K_5 or $K_{3,3}$ then G is non-planar. □

The other direction of the proof – that every non-planar graph contains a (repeated) subdivision of K_5 or $K_{3,3}$ – is done in CO 342. In fact, one can adapt the proof of Kuratowski's Theorem to obtain an algorithm which takes as input a graph G, and produces as output either a plane embedding of G or a *Kuratowski subgraph* – a subgraph of G which is a (repeated) subdivision of K_5 or $K_{3,3}$. Moreover, the running time of this algorithm is a polynomial function of the size of the input graph, and it is both practically and theoretically tractable.

Example 8.9. While we can't present planarity testing algorithms here, there is a simple heuristic which works very well for smallish graphs. Just drop the vertices one-by-one onto the plane, and at each step either adjust the partial embedding (if necessary) to accommodate the new vertex, or find a Kuratowski subgraph obstruction. Figure 8.4 shows this for the *coprimality graph*: the vertices are the integers that are at least 2, and the edges are the coprime pairs of integers. The coloured vertices indicate the two sides of a $K_{3,3}$ subgraph, so this graph is not planar.

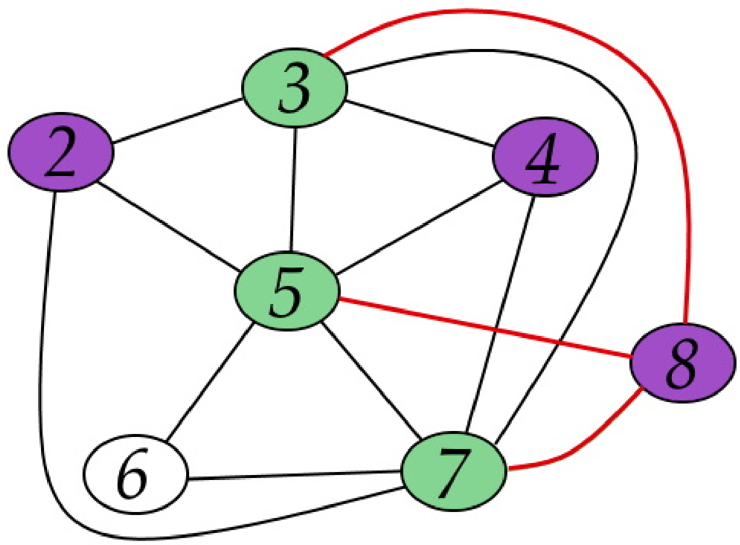


Figure 8.4: Part of the coprimality graph.

8.3 Numerology of Planar Graphs.

8.3.1 The "Faceshaking" Lemma.

Definition 8.10 (Faces of a plane embedding)**.** Let $G = (V, E, B)$ be a multigraph with a plane embedding $(\mathcal{P}, \Gamma)$.

1. The *footprint* of a subgraph H of G is the union of the points and curves representing vertices and edges in H, considered as a subset of $\mathbb{R}^2$. It is denoted by $\mathrm{fp}(H)$.
2. A *face* of $(\mathcal{P}, \Gamma)$ is a connected component of the complementary subset $\mathbb{R}^2 \setminus \mathrm{fp}(G)$ with the usual topology.
3. The *boundary* ∂F of a face $F \in \mathcal{F}$ is a subgraph of G. It has for vertices those $v \in V$ for which p_v is in the closure of F, and for edges those $e \in E$ for which γ_e is contained in the closure of F.
4. The *degree* of a face F is the number of edges plus the number of bridges of its boundary ∂F.

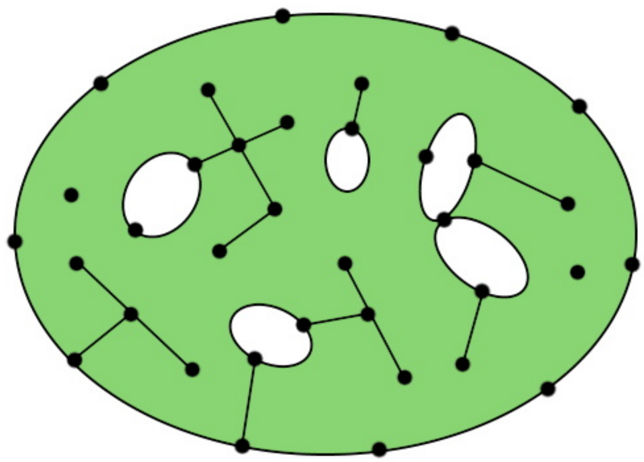


Figure 8.5: A face of an embedding, and its boundary.

(Note that the boundary of a face F is different from the boundary of a subset $S \subseteq V$ of vertices.) The footprint $\mathrm{fp}(P)$ of a path P is a simple curve, and the footprint $\mathrm{fp}(C)$ of a cycle C is a simple closed curve. The definition of the degree of a face might seem strange at first, but it is designed so that an analogue of the Handshake Lemma holds.

Example 8.11. Figure 8.5 shows a typical face in a plane embedding, and its boundary.

Lemma 8.12. *Let e be an edge of a multigraph $G = (V, E, B)$ with a plane embedding $(\mathcal{P}, \Gamma)$. Let F_1 and F_2 be the two faces on either side of the curve γ_e. Then e is a bridge if and only if $F_1 = F_2$.*

Sketch of proof. If e is not a bridge then it is contained in a cycle C in G, by Theorem 6.20. The footprint of C is a simple closed curve in $\mathbb{R}^2$, with one of the faces F_1 or F_2 in the interior and the other in the exterior. It follows from the Jordan Curve Theorem 8.3 that $F_1 \neq F_2$.

Conversely, assume that xey is a bridge in G. Let H be the connected component of G containing e, and let X and Y be the connected components of $H \smallsetminus e$ as in Proposition 6.21, with x in X and y in Y. For $q \in \mathbb{R}^2$ and (a small) $\epsilon > 0$, let $B_\epsilon(q)$ be the closed disc of radius ϵ centered at the point q. Let $B_\epsilon(X)$ be the union of the sets $B_\epsilon(q)$ for all points $q \in \mathrm{fp}(X)$ in the footprint of X. Since all the curves in the embedding are tame, we can take $\epsilon > 0$ small enough that $B_\varepsilon(X)$ is disjoint from the footprint of everything in G except the component X and the edge e. Let C be the boundary of the component of the set $\mathbb{R}^2 \smallsetminus B_\epsilon(X)$ which contains the point p_y. Again by tameness, we can take $\epsilon > 0$ sufficiently small that C is a simple closed curve which intersects γ_e at a single point and is otherwise disjoint from the footprint of G. This shows that the two faces of $(\mathcal{P}, \Gamma)$ on either side of the curve γ_e are the same face: that is, $F_1 = F_2$. $\square$

Theorem 8.13 (The "Faceshaking" Lemma). *Let $G = (V, E, B)$ be a multigraph with a plane embedding $(\mathcal{P}, \Gamma)$ and set of faces $\mathcal{F}$. Then*

$$\sum_{F \in \mathcal{F}} \deg(F) = 2\,|E|.$$

Proof. Define an auxiliary undirected multigraph H as follows. The vertices of H are the edges E and the faces $\mathcal{F}$, and $(E, \mathcal{F})$ will be a bipartition of H. Join edge $e \in E$ to face $F \in \mathcal{F}$ by an edge of H when e is in the boundary of the face F; if e is a bridge then join e to F by two edges of H. In the graph H, every "edge-vertex" has degree 2, and the degree of a "face-vertex" F is $\deg(F)$ as in Definition 8.10. The Bipartite Handshake Lemma (Exercise 5.5) holds for multigraphs and implies that

$$2\,|E(G)| = |E(H)| = \sum_{F \in F} \deg(F).$$

$\square$

8.3.2 Euler's Formula.

Theorem 8.14 (Euler's Formula). *Let $G = (V, E, B)$ be a multigraph embedded in the plane, with $|V| = n$ vertices, $|E| = m$ edges, $|\mathcal{F}| = f$ faces, and $c(G) = c$ connected components. Then*

$$n - m + f = c + 1.$$

Proof. We proceed by induction on $m = |E|$. For the basis of induction we have $m = 0$, in which case $n = c$ and $f = 1$, and the result is seen to hold. For the induction step, let G have at least one edge $e \in E$, and consider the embedding of $G' = G \smallsetminus e$ inherited from the embedding of G. Let this embedding of G' have $m' = m - 1$ edges, $n' = n$ vertices, f' faces, and c' components. By induction, we may assume that $n' - m' + f' = c' + 1$.

There are two cases: the edge e is either a bridge in G or it is not. If e is a bridge in G then $c' = c + 1$ by Corollary 6.22, and $f' = f$ by Lemma 8.12. Now

$$n - m + f = n' - (m' + 1) + f' = (n' - m' + f') - 1 = (c' + 1) - 1 = c + 1,$$

as desired. Similarly, if e is not a bridge then $c' = c$ (by definition) and $f' = f - 1$ by Lemma 8.12, and

$$n - m + f = n' - (m' + 1) + (f' + 1) = n' - m' + f' = c' + 1 = c + 1.$$

This completes the induction step, and the proof. □

Lemma 8.15 is the key point at which restriction to the case of simple graphs is important. For multigraphs with loops or multiple edges, its conclusion need not hold.

Lemma 8.15. *If G is a simple graph with at least two edges, then every face of every plane embedding of G has degree at least three.*

Proof. Exercise 8.10 □

Proposition 8.16. *Let $G = (V, E)$ be a connected planar simple graph with $|V| \geq 3$ vertices. Let n_d be the number of vertices of G degree d, for each $d \in \mathbb{N}$. (Notice that $n_0 = 0$.)*

(a) Then $5n_1 + 4n_2 + 3n_3 + 2n_4 + n_5 \geq 12 + n_7 + 2n_8 + 3n_9 + \cdots$.
(b) Equality holds in (a) if and only if, for every plane embedding of G, all faces have degree 3.

Proof. It is clear that $|V| = \sum_{d=1}^{\infty} n_d$. The Handshake Lemma implies that

$$\sum_{d=1}^{\infty} d\, n_d \;=\; \sum_{v \in V} \deg(v) \;=\; 2\,|E|.$$

(In the summations over d, only finitely many terms are non-zero.) Since G is connected with at least three vertices, Corollary 7.6 implies that G has at least two edges. Let $(\mathcal{P}, \Gamma)$ be any plane embedding of G. By Lemma 8.15, every face of this embedding has degree at least three. By the Faceshaking Lemma 8.13,

$$2\,|E| \;=\; \sum_{F\in\mathcal{F}} \deg(F) \;\geq\; 3\,|\mathcal{F}|, \tag{8.1}$$

with equality if and only if every face has degree three.

Since G is connected, Euler's Formula implies that $|V| - |E| + |\mathcal{F}| = 2$. Now multiply by 6 and calculate:

$$\begin{aligned} 12 \;&=\; 6\,|V| - 6\,|E| + 6\,|\mathcal{F}| \\ &\leq\; 6\,|V| - 6\,|E| + 4\,|E| \;=\; 6\,|V| - 2\,|E| \\ &=\; 6\sum_{d=1}^{\infty} n_d - \sum_{d=1}^{\infty} d\,n_d \;=\; \sum_{d=1}^{\infty} (6-d)\,n_d. \end{aligned}$$

Rearranging this yields part (a), and part (b) follows from the characterization of equality in (8.1). □

Corollary 8.17. *Every nonempty planar simple graph has a vertex of degree at most five.*

Proof. Consider any connected component of the graph. If it has at most two vertices then it has a vertex of degree at most one. Otherwise, Proposition 8.16 applies. Since the LHS of the inequality of part (a) is positive, the result follows. □

Example 8.18. Corollary 8.17 does not hold more generally for all planar multigraphs. Think of two vertices joined by seven edges – it has seven faces of degree two and two vertices of degree seven.

Corollary 8.19. *Let $G = (V, E)$ be a connected planar simple graph with $|V| \geq 3$ vertices. Then $|E| \leq 3\,|V| - 6$. Equality holds if and only if, for every plane embedding of G, all faces have degree 3.*

Proof. Let $(\mathcal{P}, \Gamma)$ be any plane embedding of G. Euler's Formula implies that $|V| - |E| + |\mathcal{F}| = 2$. The inequality (8.1) also holds, and we use it to eliminate $|\mathcal{F}|$ as follows:

$$6 \;=\; 3|V| - 3|E| + 3|\mathcal{F}| \;\leq\; 3|V| - 3|E| + 2|E| \;=\; 3|V| - |E|.$$

Rearranging this yields the result, with equality characterized as in (8.1). □

Example 8.20. K_5 is not planar, since it is connected with $|V| = 5$ vertices and $|E| = 10$ edges, and $3|V| - 6 = 9 < 10 = |E|$.

We have to be just a little bit more careful to show that $K_{3,3}$ is not planar, since it does satisfy the inequality $|E| = 9 \leq 12 = 3|V| - 6$.

Corollary 8.21. *Let $G = (V, E)$ be a connected planar simple graph with $|V| \geq 3$ vertices and girth $g(G) \geq 4$. Then $|E| \leq 2\,|V| - 4$. Equality holds if and only if, for every plane embedding of G, all faces have degree 4.*

Proof. Let $(\mathcal{P}, \Gamma)$ be any plane embedding of G. Euler's Formula yields $|V| - |E| + |\mathcal{F}| = 2$. Since G has no 3-cycles, every face has degree at least four. The Faceshaking Lemma then implies that $2|E| \geq 4|\mathcal{F}|$, with equality if and only if every face has degree four. Eliminating $|\mathcal{F}|$ from Euler's Formula as in the proof of Corollary 8.19 yields the result. □

Example 8.22. $K_{3,3}$ is not planar, since it is connected and has girth 4 and $2|V| - 4 = 8 < 9 = |E|$.

8.3.3 The Platonic solids.

The Platonic solids are familiar from high school geometry and have been known since antiquity. We can use graph theory to show that there are only five possible examples. (Actually constructing them as regular polyhedra in three-dimensional space is a bit of geometry beyond graph theory.)

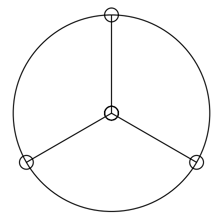


Figure 8.6: The tetrahedron.

Theorem 8.23 (The Platonic Solids)**.** *Let $d, k \geq 3$ be integers. There is a connected d-regular plane embedding that is face-k-regular in the following five cases, and in no others.*

- $(d, k) = (3, 3)$*: the tetrahedron.*
- $(d, k) = (4, 3)$*: the octahedron.*
- $(d, k) = (3, 4)$*: the cube.*
- $(d, k) = (5, 3)$*: the icosahedron.*
- $(d, k) = (3, 5)$*: the dodecahedron.*

In each case, the embedding is unique up to isomorphism.

Proof. Let such an embedding have n vertices, m edges, f faces, and $c = 1$ component. By the Handshake and Faceshaking Lemmas, we have $dn = 2m = kf$. Now substitute $n = 2m/d$ and $f = 2m/k$ into Euler's Formula: $n-m+f = 2$. The condition is equivalent to $1/d+1/k = 1/2+1/m$. The five pairs listed are the only ones for which $1/d + 1/k > 1/2$, as one can easily check. Figures 8.6 to 8.10 show the five Platonic solids as plane embeddings of graphs. Uniqueness up to isomorphism is left as Exercise 8.14. □

The case $d = 2$ and $k \geq 3$ is satisfied by a plane embedding of a k-cycle C_k. The case $d \geq 3$ and $k = 2$ can be realized if one allows multigraphs in place of graphs. If one allows infinite graphs then examples for all $d \geq 3$ and $k \geq 3$ can be constructed. Only three of these cases have nice embeddings in Euclidean space, though. These "regular tesselations of the plane" are considered in Section **??**.

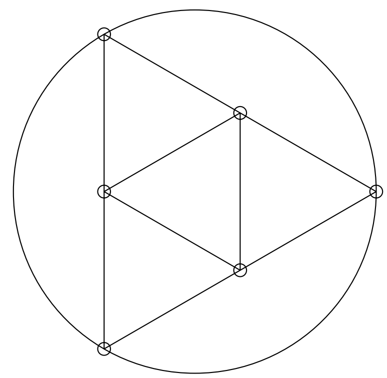


Figure 8.7: The octahedron.

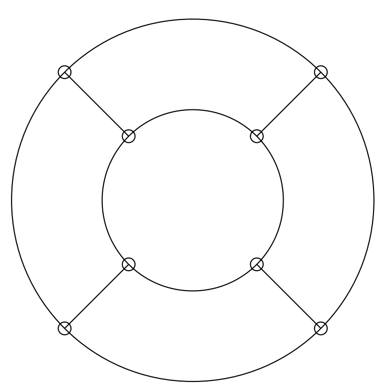


Figure 8.8: The cube.

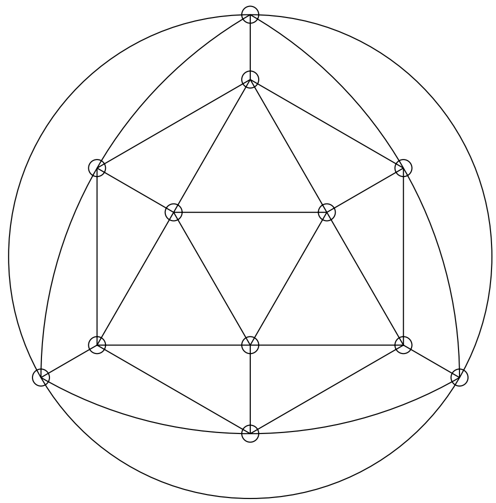


Figure 8.9: The icosahedron.

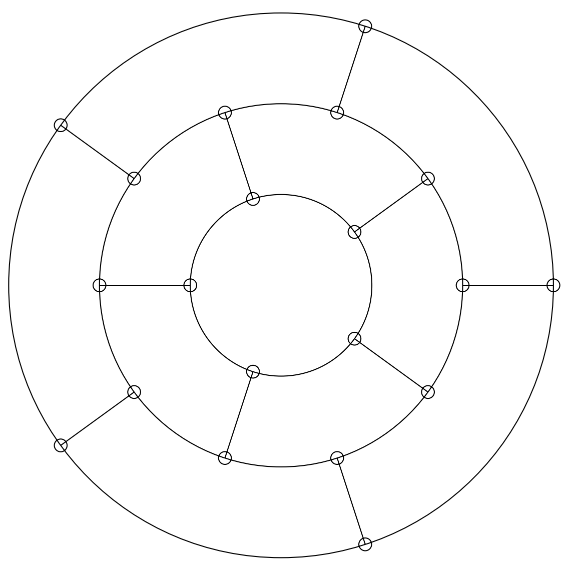


Figure 8.10: The dodecahedron.

8.4 Planar Duality.

Definition 8.24 (Dual Graph)**.** Let $(\mathcal{P}, \Gamma)$ be a plane embedding of a multigraph $G = (V, E, B)$, with set of faces $\mathcal{F}$. The *dual graph* of this embedding is a multigraph $G^* = (V^*, E^*, B^*)$. The vertices of G^* are the faces of G: that is, $V^* = \mathcal{F}$. The edges of G^* correspond bijectively to the edges of G: that is, $E^* = \{e^* : \ e \in E\}$. For every bridge e of G, e^* is a loop in G^* at the vertex corresponding to the face in $\mathcal{F}$ that has e in its boundary. For every non-bridge e of G, e^* is an edge incident with the two faces in $\mathcal{F}$ that have e in their boundary.

Lemma 8.25. *Let $(\mathcal{P}, \Gamma)$ be a plane embedding of a multigraph $G = (V, E, B)$, with set of faces $\mathcal{F}$. Let $G^* = (V^*, E^*, B^*)$ be the dual graph of this embedding. For every face $F \in \mathcal{F}$, the degree of F as a face of $(\mathcal{P}, \Gamma)$ equals the degree of F as a vertex of G^*.*

Proof. Exercise 8.17. □

Lemma 8.25 shows that the Faceshaking Lemma for a plane embedding is equivalent to the Handshake Lemma for its dual graph.

Example 8.26. Figure 8.11 shows two plane embeddings of isomorphic graphs. The dual graphs of these embeddings have different degree sequences – one is $(6\ 4\ 3\ 3)$ and the other is $(5\ 5\ 3\ 3)$. While the graphs are isomorphic, the dual graphs of the plane embeddings are not isomorphic.

To define the dual embedding of a plane embedding of a graph, we need the following lemma.

Lemma 8.27. *Let F be a face of a plane embedding $(\mathcal{P}, \Gamma)$ of a multigraph $G = (V, E, B)$. Let p^* be any point in F. For each edge $e \in E(\partial F)$ in the boundary of F, let $q_e = \gamma_e(1/2)$ be a midpoint of the curve representing e. Let $\mathcal{Q} = \{q_e : \ e \in E(\partial F)\}$. Then there is a set $\mathcal{C}$ of simple curves in $\mathbb{R}^2$ with the following properties.*

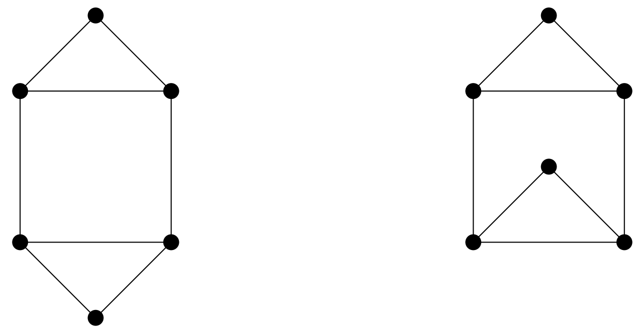


Figure 8.11: Two different plane embeddings of the same graph.

1. *Each simple curve in $\mathcal{C}$ has endpoints p^* and q_e for some $q_e \in \mathcal{Q}$, and is contained in the set $F \cup \{q_e\}$.*
2. *If $e \in E(\partial F)$ is not a bridge then there is exactly one such curve in $\mathcal{C}$.*
3. *If e is a bridge then there are exactly two such curves in $\mathcal{C}$, the union of which is a simple closed curve that separates the endpoints of γ_e.*
4. *The curves in $\mathcal{C}$ are pairwise disjoint except at their endpoints.*

Proof of Lemma 8.27 is a rather tedious exercise in point-set topology – not graph theory – and is omitted. (It is intuitively clear in the case that ∂F is a single cycle. By induction on the number of components of ∂F, it then follows in the case that ∂F has no bridges. Finally, the general case follows by induction on the number of bridges. Proving Exercise 8.12 first helps to clarify the picture.)

Definition 8.28 (Dual Embedding)**.** Let $(\mathcal{P}, \Gamma)$ be a plane embedding of a multigraph $G = (V, E, B)$, with set of faces $\mathcal{F}$. Let $G^* = (V^*, E^*, B^*)$ be the dual graph of this embedding. We define a plane embedding $(\mathcal{P}^*, \Gamma^*)$ of G^* as follows.

UNDER CONSTRUCTION

UNDER CONSTRUCTION

Example 8.29. Ponder Figure 8.12.

Proposition 8.30. *If G is a connected multigraph embedded in the plane, then G^{**} is isomorphic with G.*

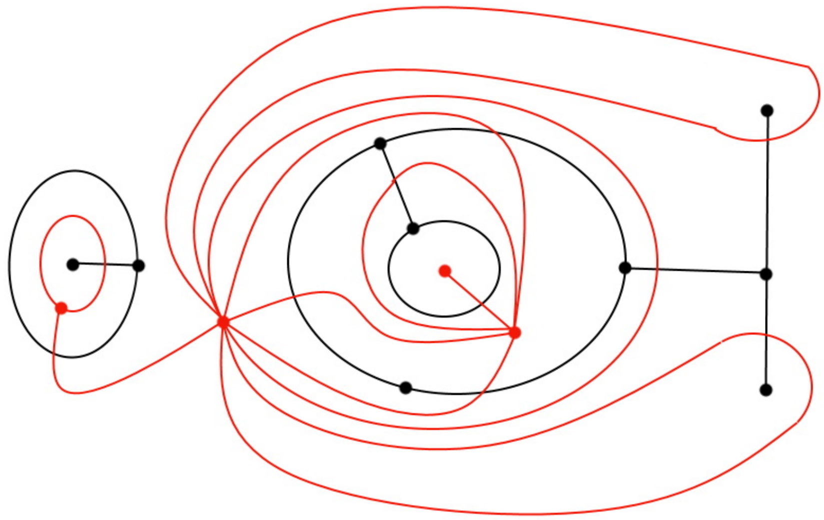


Figure 8.12: A plane embedding (black) and its dual embedding (red).

Proof. UNDER CONSTRUCTION □

8.5 Exercises.

Exercise 8.1.

(a) Finish the proof of Lemma 8.5.
(b) Finish the proof of Lemma 8.7.

Exercise 8.2. For each of the graphs in Figure 8.13, give either a plane embedding or a Kuratowski subgraph.

Exercise 8.3. For each of the graphs in Figure 8.14, give either a plane embedding or a Kuratowski subgraph.

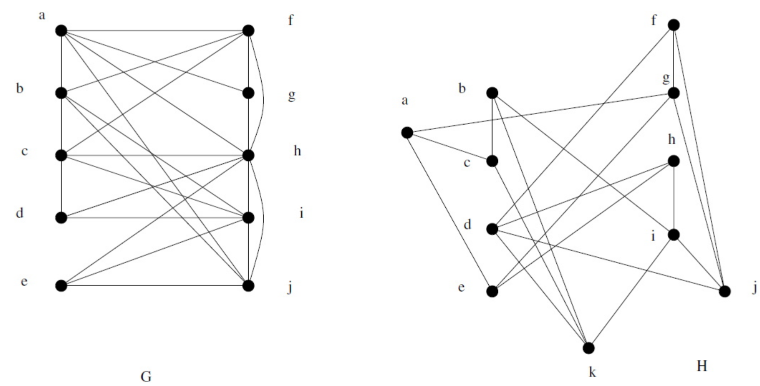


Figure 8.13: Examples for Exercise 8.2.

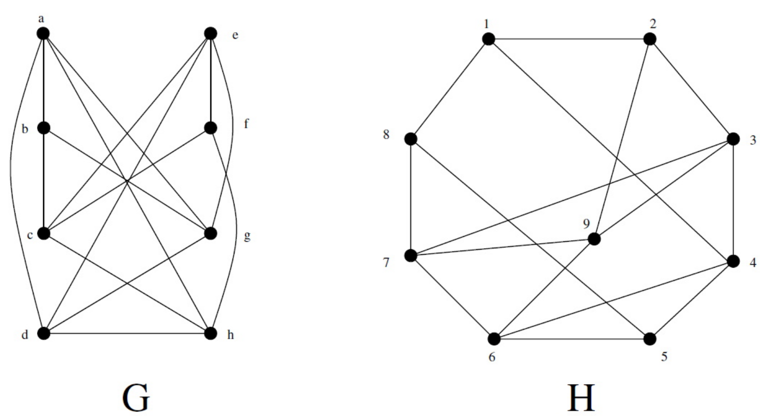


Figure 8.14: Examples for Exercise 8.3.

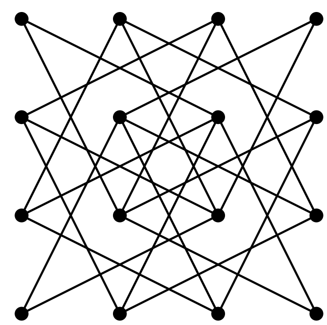


Figure 8.15: The Knight's move graph $\mathrm{KM}(4,4)$.

Exercise 8.4. For $k \geq 2$ and $n \geq 2k$, determine whether or not the circulant $C_n(1,k)$ is planar.

Exercise 8.5. For $k \geq 3$ and $n \geq 2k$, determine whether or not the circulant $C_n(2,k)$ is planar.

Exercise 8.6. For $k \geq 4$ and $n \geq 2k$, determine whether or not the circulant $C_n(3,k)$ is planar.

Exercise 8.7. For $a, b \in \mathbb{N}$, the *a-by-b knight's move graph* $\mathrm{KM}(a,b)$ is defined as follows. The vertex-set is $V(\mathrm{KM}(a,b)) = \{1,2,...,a\} \times \{1,2,...,b\}$. Two vertices (v_1,v_2) and (w_1,w_2) are adjacent if and only if

$$(v_1 - w_1)^2 + (v_2 - w_2)^2 = 5.$$

Figure 8.15 is a picture of $\mathrm{KM}(4,4)$.

(a) Show that $\mathrm{KM}(4,4)$ is not a planar graph.
(b) There is a positive integer k such that $\mathrm{KM}(k,3)$ is planar, but $\mathrm{KM}(k+1,3)$ is not planar. Determine the value of k and give a complete explanation.

Exercise 8.8. Give an example of a planar bipartite 3-regular simple graph with $|V| = 14$ vertices.

Exercise 8.9. Show that if G is planar and 3-regular then the line-graph $L(G)$ of G is planar.

Exercise 8.10. Prove that if G is simple and has at least two edges then every face of every plane embedding of G has degree at least three (Lemma 8.15).

Exercise 8.11. Give an example of a plane embedding of a simple graph G such that: every vertex has degree at least three; every face has degree at least four; G contains a 3-cycle.

Exercise 8.12. Let F be a face of a plane embedding of a multigraph $G = (V, E, B)$.

(a) Every edge $e \in E(\partial F)$ in the boundary of F is contained in at most one cycle in ∂F.
(b) The boundary of F is an edge-disjoint union of cycles, bridges, and vertices of degree zero.
(c) If G contains a cycle then ∂F contains a cycle.

Exercise 8.13. Let $G = (V, E, B)$ be a multigraph with a plane embedding $(\mathcal{P}, \Gamma)$.

(a) Show that all faces in $\mathcal{F}$ have even degree if and only if G is bipartite.
(b) Show that all vertices in V have even degree if and only if the dual graph G^* is bipartite.

Exercise 8.14. Show that each of the five Platonic solids are unique up to isomorphism.

Exercise 8.15. Let $G = (V, E)$ be a 3-regular connected simple graph embedded in the plane, with f_4 faces of degree 4 and f_7 faces of degree 7, and no faces of any other degree. The cube is an example with

$f_4 = 6$ and $f_7 = 0$.

(a) Show that for any such graph there is an $a \in \mathbb{N}$ such that $f_4 = 6 + a$ and $f_7 = 2a$.
(b) Give an example of such a graph with $f_7 = 2$.
(c) Give an example of such a graph with $f_7 = 4$.
(d)* (Not required.) Give examples of such graphs for all $a \in \mathbb{N}$.

Exercise 8.16. Let $G = (V, E)$ be a 4-regular connected graph embedded in the plane, with f_d faces of degree $d \in \{3, 4, 6\}$ and no faces of any other degree. The octahedron is an example with $f_3 = 8$ and $f_4 = f_6 = 0$.

(a) Show that for any such graph, $f_3 = 8 + 2f_6$.
(b) Give an example of such a graph with $f_3 = 8$ and $f_4 = 6$.
(c) Give an example of such a graph with $f_4 = 0$ and $f_6 = 2$.
(d) Show that there is no such graph with $f_4 = 0$ and $f_6 = 1$.
(e) Give an example of such a graph with $f_6 \geq 3$.
(f)* (Not required.) Give examples of such graphs for as many values of $f_6 \in \mathbb{N}$ as you can.

Exercise 8.17. Prove Lemma 8.25.

Exercise 8.18. Let $G = (V, E, B)$ be a connected multigraph with a plane embedding $(\mathcal{P}, \Gamma)$ and dual graph $G^* = (V^*, E^*, B^*)$. Let (V, T) be a spanning tree of G. Show that $(V^*, \{e^* \in E^* : e \notin T\})$ is a spanning tree of G^*.

Chapter 9

Graph Colouring.

Example 9.1 (The Channel Assignment Problem)**.** A commercial radio station broadcasts with a power of several tens of kilowatts, and its signal can be received from a distance of up to several tens of kilometers. To avoid interference between the signals, if two transmitters are close enough and powerful enough that they could both be received from the same location, then they must broadcast at different frequencies. The range of available frequencies is split into several channels. Radio bandwidth is a valuable commodity, so one would like to minimize the total number of channels that are assigned to the radio stations in a given region.

The channel assignment problem is a prototypical example of a graph colouring problem. Let the vertices be the radio transmitters. Join two vertices by an edge if the corresponding transmitters would interfere with one another (if they broadcast at the same frequency). We are motivated to find a function from vertices to channels such that vertices joined by an edge get different channels, and the total number of channels is minimized.

9.1 Chromatic Number.

Definition 9.2 (Proper colouring, chromatic number)**.** Let $G = (V, E)$ be a graph. Let X be a finite set of *colours*. (Often, $X = \{1, 2, ..., k\}$ for some $k \in \mathbb{N}$.)

1. A *(proper) X-colouring* of G is a function $f : V \to X$ such that if $vw \in E$ then $f(v) \neq f(w)$.
2. If $|X| = k$ then such a function f is a *(proper) k-colouring* of G.
3. The *chromatic number* $\chi(G)$ of G is the smallest natural number $k \in \mathbb{N}$ for which G has a (proper) k-colouring.

Example 9.3. In Definition 9.2, we can take $X = V$ for the set of colours, and then the identity function $\iota : V \to V$ is a proper $|V|$-colouring of G. This shows that the chromatic number exists and that $\chi(G) \leq |V|$. It is easy to see that for complete graphs, $\chi(K_n) = n$. The converse is also true: if $\chi(G) = |V(G)|$, then G is a complete graph.

Example 9.4. Graphs with small chromatic number are easy to understand.

- The only graph with chromatic number zero is the empty graph.
- A graph has chromatic number one if and only if it has no edges and at least one vertex.
- A graph has chromatic number two if and only if it is bipartite and has at least one edge.

Graphs with chromatic number three or more are hard to understand. Consider the following problem: given a graph G, determine whether or not $\chi(G) = 3$. Nobody has a good ("polynomial time") algorithm to solve this problem. It is known to be what is called an "NP-complete" problem. Roughly speaking, this means that if there is a good algorithm to solve this problem then there are good algorithms to solve a huge variety of problems, all of which are considered to be hard. In short, this is considered to be a hard problem.

The inequality of Example 9.3 is terrible, and can easily be improved.

Proposition 9.5. *Let G be a graph and let $d_{\max}(G)$ be the maximum degree of a vertex in G. Then $\chi(G) \leq 1 + d_{\max}(G)$.*

Proof. Let $k = 1 + d_{\max}(G)$ and $X = \{1, 2, ..., k\}$. We construct a proper k-

colouring $f : V \to X$ of G as follows. List the vertices of G as $v_1, v_2, ..., v_n$ in any order. Define $f(v_i) = i$ for all $1 \le i \le k$. There are no edges violating the condition of being a proper colouring at this stage. Now, inductively, assume that we have properly coloured vertices $v_1, ..., v_{j-1}$ for some $2 \le j \le n$. The degree of v_j is at most $k-1$, so v_j has at most $k-1$ neighbours among the previously coloured vertices $v_1, ..., v_{j-1}$. Thus, at most $k-1$ colours are used on these previously coloured neighbours of v_j. Since there are k colours available, there is a colour we can choose for $f(v_j)$ so that no edge gets the same colour at both ends. Now we have properly coloured the vertices $v_1, ..., v_j$. By induction, we arrive at a proper k-colouring of G. □

Example 9.6. Consider the infinite graph with vertex-set $\mathbb{R}^2$, with an edge between two vertices if and only if the distance between the two points is exactly one. Figure 9.1 shows a 7-colouring of this graph. (The colours on the boundaries of the hexagons can be defined in a way to make this work.) If the distance between the centers of adjacent hexagons is λ, then the lengths of the edges is $\lambda/\sqrt{3}$, and the diameter of the hexagons is $2\lambda/\sqrt{3}$. Look at the hexagons of any one of the seven colours. The distance between corners of different hexagons of the same colour is at least $(\lambda/\sqrt{3})|2 + \mathrm{e}^{\mathrm{i}\pi/3}| = \lambda\sqrt{7/3}$. So there will be no monochromatic edges if $2\lambda/\sqrt{3} < 1$ and $\lambda\sqrt{7/3} > 1$. So if Figure 9.1 is scaled by any factor $0.6546 \approx \sqrt{3/7} < \lambda < \sqrt{3/4} \approx 0.8660$ then it gives a proper 7-colouring of $\mathbb{R}^2$.

Also shown: the graph of Figure 9.2 to appropriate scale, a random walk in the unit distance graph of $\mathbb{R}^2$, and an embedding of the complete graph K_7 on the torus.

Example 9.7 (The Hadwiger-Nelson Problem)**.**
What is the chromatic number of the unit distance graph of $\mathbb{R}^2$?
Example 9.6 shows that it is at most seven, and the graph of Figure 9.2 shows that it is at least four – this has been known since 1950. In 2018, de Grey – an amateur mathematician – constructed an example with chromatic number five. His first example had more than 20,000 vertices – this has since been reduced to around 850. *Quanta* magazine published a nice non-technical article about this.

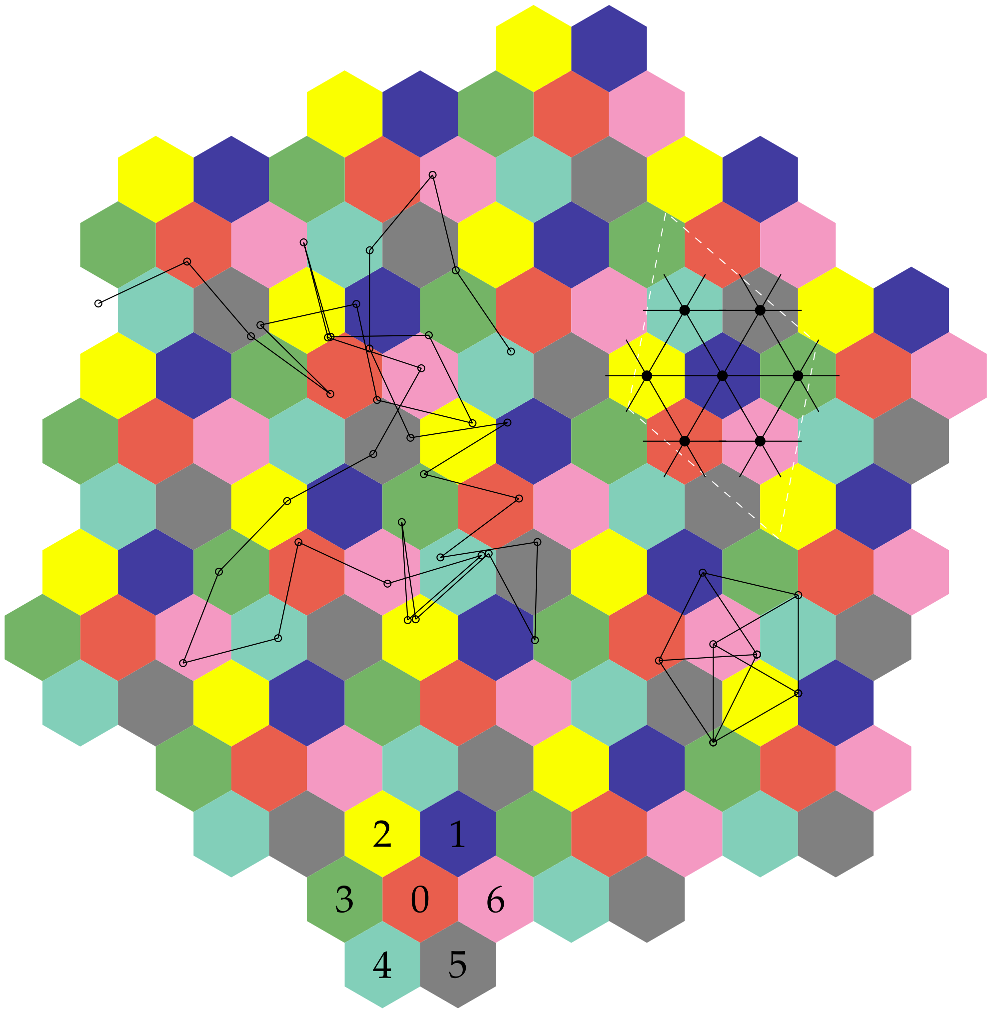


Figure 9.1: A 7-colouring of the unit-distance graph of $\mathbb{R}^2$.

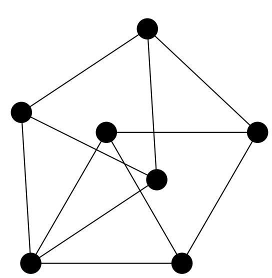


Figure 9.2: A 4-chromatic unit distance graph.

9.2 Colouring Planar Graphs.

9.2.1 The Four Colour Theorem.

In 1852, an engineer named Guthrie was looking at a map of England, and noticed that he could colour the counties in such a way that those which shared a border had different colours, and that he could do this using only four colours. He wondered whether this was possible for any map drawn on the plane (or sphere). This became the Four Colour Conjecture or Four Colour Problem, and has motivated a huge amount of graph theory and combinatorics since then. It was finally solved in 1976 by Appel and Haken, with a then-controversial computer-aided proof.

Theorem 9.8 (The Four Colour Theorem)**.** *Every planar graph can be properly coloured with at most four colours.*

Notice that Guthrie was thinking of colouring the faces of an embedded plane graph in such a way as to give a proper colouring of the dual graph of the embedding. Since every connected planar graph is the dual of some embedded plane graph, the problems of colouring vertices or of colouring faces are equivalent.

Here is a link to an amusing puzzle based on the Four Colour Theorem.

Even the shortest proofs of the Four Colour Theorem still involve a long computer-aided case analysis. But we can do fairly well with a little bit of effort.

9.2.2 The Five Colour Theorem.

As an easy warm-up exercise, we can colour planar graphs with at most six colours.

Proposition 9.9 (The Six Colour Theorem)**.** *Every planar graph can be properly coloured with at most six colours.*

Proof. Let $G = (V, E)$ be a planar graph. We prove the result by induction on $|V| = n$, the basis $n \leq 6$ being evident. Assume that $n \geq 7$. By Corollary 8.17, G has a vertex v of degree at most five. Since $G \setminus v$ is planar, by induction we have a proper 6-colouring $f : V(G \setminus v) \to \{1, 2, ..., 6\}$ of $G \setminus v$. At most five colours are used by f on the neighbours of v, so at least one colour is available to use for $f(v)$ without producing edges with the same colour at both ends. So G has a proper 6-colouring, completing the induction step, and the proof. □

The idea behind the proof of the Six Colour Theorem is good – perhaps with a little more care in the induction step we can reduce the number of colours required. In 1879, Kempe thought he had solved the Four Colour Problem along these lines. In 1890, Heawood found a flaw in Kempe's argument – but he also saw that it worked for five colours.

Theorem 9.10 (The Five Colour Theorem)**.** *Every planar graph can be properly coloured with at most five colours.*

Proof. Let $G = (V, E)$ be a planar graph. We prove the result by induction on $|V| = n$, the basis $n \leq 5$ being evident. Assume that $n \geq 6$. By Corollary 8.17, G has a vertex v of degree at most five. Since $G \setminus v$ is planar, by induction we have a proper 5-colouring $f : V(G \setminus v) \to \{1, 2, ..., 5\}$ of $G \setminus v$.

If at most four colours are used by f on the neighbours of v, then at least one colour is available to use for $f(v)$ without producing edges with the same colour at both ends. In this case G has a proper 5-colouring, as desired. In the remaining case, f uses all five colours on the neighbours of v. In particular, v has degree five.

Now embed G in the plane, with the neighbours of v labelled $w_1, w_2, ..., w_5$ in cyclic order around v. We may permute the colours to assume that $f : V \setminus \{v\} \to \{1, 2, ..., 5\}$ is such that $f(w_i) = i$ for all $1 \leq i \leq 5$. The idea is to adjust the 5-colouring f so as to produce another 5-colouring which uses only four colours on the neighbours of v. Then the previous argument can be applied to finish the induction step.

For $i \in \{1, 2, ..., 5\}$, let V_i be the set of vertices $z \in V$ such that $f(z) = i$. For $1 \leq i < j \leq 5$, let G_{ij} be the subgraph of $G \setminus v$ induced by $V_i \cup V_j$. Let H be a connected component of G_{ij}. Define a new 5-colouring g of $G \setminus v$

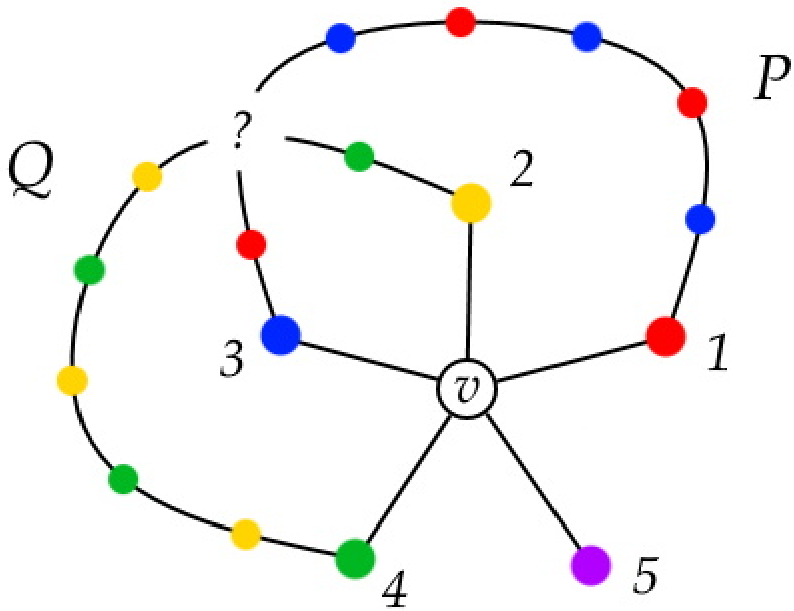


Figure 9.3: Proof of the Five Colour Theorem.

as follows. Consider any $z \in V(G \setminus v)$: if $z \notin V(H)$ then let $g(z) = f(z)$; if $z \in V(H)$ then let $g(z) = i$ if $z \in V_j$, and let $g(z) = j$ if $z \in V_i$. Notice that this g is also a proper 5-colouring of $G \setminus v$. We refer to this operation as "flipping f along H".

Now consider the subgraph G_{13} of $G \setminus v$, and the component H of G_{13} that contains w_1. If w_3 is not in H then consider the result g of flipping f along H. This is a proper 5-colouring of $G \setminus v$ such that $g(w_1) = g(w_3) = 3$. Thus, g uses only four colours on the neighbours of v, and we have reduced the problem to a previously solved case. In the remaining case, there is a path P in G_{13} from w_1 to w_3.

Next, consider the subgraph G_{24} of $G \setminus v$, and the component H of G_{24} that contains w_2. If w_4 is not in H then consider the result g of flipping f along H. This is a proper 5-colouring of $G \setminus v$ such that $g(w_2) = g(w_4) = 4$. Thus, g uses only four colours on the neighbours of v, and we have reduced the problem to a previously solved case. In the remaining case, there is a path Q in G_{24} from w_2 to w_4.

Now, in the embedding of G, the footprint of the cycle $(v\, w_1\, P\, w_3\, v)$ is a simple closed curve in $\mathbb{R}^2$ that separates the points representing w_2 and w_4. Therefore, the footprints of the paths P and Q have a point in common. (See Figure 9.3.) Since G is embedded properly, this common point must represent a common vertex $z \in V(P) \cap V(Q)$. But $V(G_{13}) \cap V(G_{24}) = \varnothing$, by the way these subgraphs are defined. This contradiction shows that the paths P and Q cannot both exist. This completes the induction step, and the proof. □

9.3 Chromatic Number versus Girth.

A large chromatic number seems to require a graph to have highly complicated connections among its vertices. Large girth seems to require very sparse connections. One might expect some tension between these properties – that is the case, but somewhat surprisingly there are graphs with arbitrarily large girth and chromatic number at the same time.

Theorem 9.11 (Erdős, 1959). *For all $k \geq 2$ and $g \geq 3$, there is a graph with girth at least g and chromatic number at least k.*

You might see a proof of Theorem 9.11 in a fourth-year course on graph theory. We will find graphs of girth four and any chromatic number.

Definition 9.12. . Let $G = (V, E)$ be a graph. Let $V' = \{v' : v \in V\}$ be a set disjoint from V, let z be another vertex not in $V \cup V'$, and let $\mathcal{M}(G)$ be the graph with vertices $V(\mathcal{M}(G)) = V \cup V' \cup \{z\}$and edges $E(\mathcal{M}(G)) = E(G) \cup \{zv' : v \in V\} \cup \{v'w : v \in V$ and $vw \in E\}$.
This is called the *Mycielski construction*.

Lemma 9.13. *If G has girth at least four then $\mathcal{M}(G)$ has girth at least four.*

Proof. Exercise 9.7. □

Proposition 9.14. *For any graph G, $\chi(\mathcal{M}(G)) = 1 + \chi(G)$.*

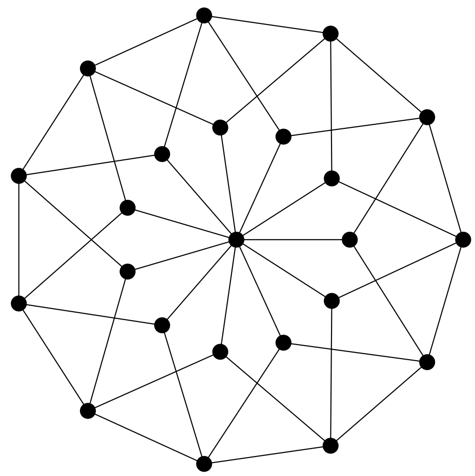


Figure 9.4: The Mycielski construction $\mathcal{M}(C_{11})$.

Proof. Exercise 9.8. □

Corollary 9.15. *For every $k \geq 2$ there exists a k-chromatic graph of girth four.*

Proof. Examples for $k = 2$ and $k = 3$ are easily found. By Lemma 9.13 and Proposition 9.14, the sequence

$$\mathcal{M}(C_5),\ \mathcal{M}(\mathcal{M}(C_5)),\ \mathcal{M}(\mathcal{M}(\mathcal{M}(C_5))),\ \ldots$$

settles the remaining cases. □

9.4 Exercises.

Exercise 9.1. Let G be a graph for which $\chi(G) = |V(G)|$. Prove that G is a complete graph.

Exercise 9.2. Prove that if G is a self-complementary graph then $\chi(G) \geq |V(G)|^{1/2}$.

Exercise 9.3. Fix a natural number $k \in \mathbb{N}$. A graph $G = (V, E)$ is *k-degenerate* if its vertices can be listed $v_1, \ldots, v_n$ with the following property: for all $1 \leq j \leq n$, the vertex v_j has at most k neighbours among $v_1, \ldots, v_{j-1}$.

(a) If G is k-degenerate then $\chi(G) \leq 1 + k$.
(b) If G is planar then G is 5-degenerate.
(c) If G is planar and has no 3-cycles then G is 3-degenerate.

Exercise 9.4. Show that a graph with $2k \geq 2$ vertices and $k^2 + 1$ edges is not 2-colourable.

Exercise 9.5. Show that a planar graph with $n \geq 3$ vertices and $m = 2n - 3$ edges is not 2-colourable.

Exercise 9.6. Let G be a 3-regular planar graph with its edges partitioned into three perfect matchings (see Example 10.3). Show that the dual of any plane embedding of G is 4-colourable.

Exercise 9.7. Prove Lemma 9.13

Exercise 9.8.

(a) Show that the chromatic number of $\mathcal{M}(C_5)$ is 4.
(b) Show that if G is k-colourable then $\mathcal{M}(G)$ is $(k+1)$-colourable.
(c) Show that if $\mathcal{M}(G)$ is k-colourable then $\mathcal{M}(G)$ has a k-colouring $f : V(\mathcal{M}(G)) \to \{1, 2, \ldots, k\}$ such that $f^{-1}(k) = \{z\}$. (That is, z is the only vertex of $\mathcal{M}(G)$ at which f takes the value k.)
(d) Prove Proposition 9.14.

Chapter 10

Bipartite Matching.

Example 10.1 (The Job Assignment Problem)**.** Shawn has contracts to renovate some residential properties in Kitchener. Each property needs at least one of several types of jobs done: carpentry, plumbing, electrical, drywall, painting, *et cetera*. Shawn is on good terms with several work teams in the region. Not every team is proficient at each kind of job. Shawn would like to assign jobs to teams so that as many teams as possible are working, and all are working on jobs they are good at. Each team can only work on one job. Each job can only be assigned to one team. Is there a method to solve this kind of problem? In particular, is the assignment of jobs to teams in the graph of Figure 10.1 as good as possible? (Work teams are denoted by letters, jobs are denoted by numbers, and the edge A1 means that team A is qualified to do job 1.)

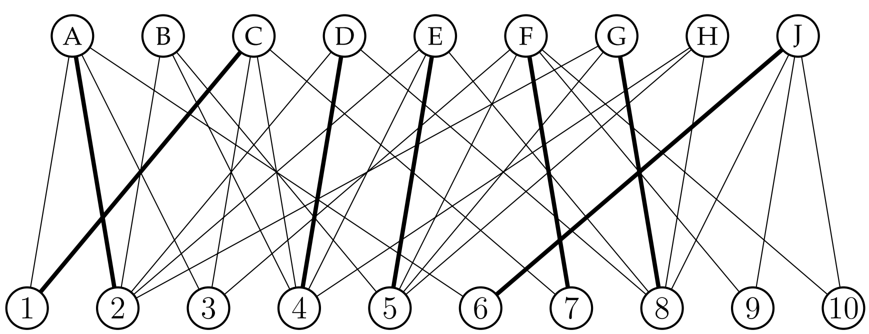


Figure 10.1: An instance of the job assignment problem

10.1 Matchings and Covers.

Definition 10.2 (Matchings)**.** Let $G = (V, E)$ be a graph.

1. A *matching* in G is a subset $M \subseteq E$ of edges such that in the spanning subgraph (V, M), every vertex has degree at most one.
2. A vertex is M*-saturated* if it has degree one in (V, M), and is M-*unsaturated* if it has degree zero in (V, M).
3. A matching M is *maximum* if $|M|$ is as big as possible among all matchings of G.

Example 10.3 (The Game of Slither)**.** The game of Slither can be played on any graph; for concreteness, imagine a grid like $P_5 \,\square\, P_5$. There are two players who take alternate turns – the last player to be able to move wins. The first player chooses any edge of G to form a path with two vertices. Next, the second player chooses an edge to extend this path (at either end) if possible, or else loses. The chosen subgraph must remain a path. Next, play returns to the first player, who must also either extend the path or lose, and so it continues.

A matching M is *perfect* if every vertex of G is saturated. In Exercise 10.4, you are asked to show that if G has a perfect matching then the first player has a winning strategy for Slither.

Definition 10.4 (Alternating Paths, Augmenting Paths)**.** Let $G = (V, E)$ be a graph, and let M be a matching in G.

1. A path $P = (v_0\, v_1 \dots v_k)$ in G is M*-alternating* when:
   for all $1 \leq i \leq k - 1$, $v_{i-1}v_i \in M$ if and only if $v_i v_{i+1} \notin M$.
2. A path in G is M*-augmenting* if it is M-alternating, has at least one edge, and both its ends are M-unsaturated.

Informally, a path is M-alternating when "every second edge is in M".

Example 10.5. If G has an M-alternating path P then we can find another matching, as follows. We "flip" the edges on the path P: those that were in M come out, and those that were not in M go in. This is the symmetric difference of sets $M' = M \triangle E(P)$. Since M is a matching and P is an M-alternating path, it follows that M' is also a matching. Notice that if P is an M-augmenting path then $|M'| = 1 + |M|$.

For example, $(\text{B}\,2\,\text{A}\,6\,\text{J}\,9)$ is an augmenting path for the matching in Figure 10.1, so that matching can be improved.

Proposition 10.6. *Let $G = (V, E)$ be a graph and $M \subseteq E$ a matching in G. Then M is a maximum matching if and only if there are no M-augmenting paths.*

Proof. One direction follows directly from Example 10.5: if M has an augmenting path P then $M' = M \triangle E(P)$ is a matching in G that is strictly bigger than M. Thus, if M is a maximum matching then there are no M-augmenting paths.

Conversely, assume that M is not a maximum matching. There is a maximum matching $\widetilde{M}$ of G, and $|M| < |\widetilde{M}|$. Now consider the spanning subgraph $H = (V, M \cup \widetilde{M})$ of G. Since H is the union of two matchings, every vertex has degree at most two. So every connected component of H is either a path or a cycle. Moreover, each path component is both M-alternating and $\widetilde{M}$-alternating, and each cycle component has edges alternating between M and $\widetilde{M}$ as well. In particular, each cycle component has the same number of edges in M as in $\widetilde{M}$. Now, since $|\widetilde{M}| > |M|$, there must be some component of H with more edges in $\widetilde{M}$ than in M. By the above, this is an M-augmenting path. $\square$

We next come to a concept which is dual to the idea of a matching. It will be useful for characterizing maximum matchings in bipartite graphs.

Definition 10.7 (Covers)**.** Let $G = (V, E)$ be a graph.

1. A *cover* in G is a subset $C \subseteq V$ of vertices such that every edge has at least one end in C: for all $e \in E$, $e \cap C \neq \varnothing$.
2. A cover C in G is *minimum* if $|C|$ is as small as possible among all covers of G.

Proposition 10.8. *Let $G = (V, E)$ be a graph, let $M \subseteq E$ be a matching in G, and let $C \subseteq V$ be a cover in G.*

(a) Then $|M| \leq |C|$.
(b) If $|M| = |C|$ then M is maximum and C is minimum.

Proof. Let X be the set of pairs (v, e) in $C \times M$ for which $v \in e$. Since M is a matching it follows that

$$|X| = \sum_{(v,e)\in X} 1 = \sum_{v\in C} \deg_M(v) \leq \sum_{v\in C} 1 = |C|.$$

Since C is a cover it follows that

$$|X| = \sum_{(v,e)\in X} 1 = \sum_{e\in M} |e \cap C| \geq \sum_{e\in M} 1 = |M|.$$

This proves (a). For part (b), let matching M and cover C satisfy $|M| = |C|$. If M' is any other matching then, by part (a), $|M'| \leq |C| = |M|$. Thus, M is a maximum matching. The proof that C is a minimum cover is similar. □

Example 10.9. Notice that for an odd cycle C_{2k+1}, a maximum matching has k edges and a minimum cover has $k + 1$ vertices. So there can be a gap between the two sides in the inequality of Proposition 10.8. Taking a disjoint union of many odd cycles, one sees that this gap can be arbitrarily large. It is interesting to try to find connected examples with arbitrarily large "gap" $\min |C| - \max |M|$. (See Exercise 10.6.)

10.2 König's Theorem.

In Shawn's job assignment problem (Example 10.1) the underlying graph was bipartite – does that help? It turns out that – yes – it does help. König's Theorem states that for bipartite graphs there is no "gap" as in Example 10.9.

Theorem 10.10 (König's Theorem, 1931)**.** *Let $G = (V, E)$ be a graph with a bipartition (A, B). Let $M \subseteq E$ be a maximum matching and let $C \subseteq V$ be a minimum cover in G. Then $|M| = |C|$.*

We prove this after a close look at a matching in a bipartite graph.

10.2.1 Anatomy of a bipartite matching.

The key step is to examine the relations between matchings and alternating paths.

Definition 10.11. Let $G = (V, E)$ be a graph with a bipartition (A, B). Let $M \subseteq E$ be a matching in G.

1. Let X_0 be the set of M-unsaturated vertices in A.
2. Let Z be the set of vertices in V that are reachable from a vertex in X_0 by an M-alternating path.
3. Let $X = Z \cap A$ and $Y = Z \cap B$.

Proposition 10.12. *Let $G = (V, E)$ be a graph with a bipartition (A, B). Let $M \subseteq E$ be a matching in G. Define the subsets X and Y as in Definition 10.11.*

(a) If $y \in Y$ is unsaturated then G has an M-augmenting path.
(b) There are no edges of G between X and $B \setminus Y$.
(c) There are no edges of M between Y and $A \setminus X$.

Proof. For part (a), let $P = (v_0\, v_1 \ldots v_k)$ be an M-alternating path with $v_0 \in X_0$ and $v_k = y \in Y$. If y is M-unsaturated then P is an M-augmenting path.

For part (b), suppose that $e = xb \in E$ is an edge with $x \in X$ and $b \in B \setminus Y$. Then there is an M-alternating path $(v_0\, v_1 \ldots v_k)$ with $v_0 \in X_0$ and $v_k = x \in A$. since $v_0 \in A$ and $v_k \in A$ and G is bipartite, it follows that k is even. Now, since v_0 is M-unsaturated and the path is M-alternating and k is even, it follows that $v_{k-1}v_k \in M$ is a matching edge. Thus, $v_k b \notin M$ is not a matching edge. But now $(v_0\, v_1 \ldots v_k\, b)$ is an M-alternating path, so that $b \in Y$, a contradiction.

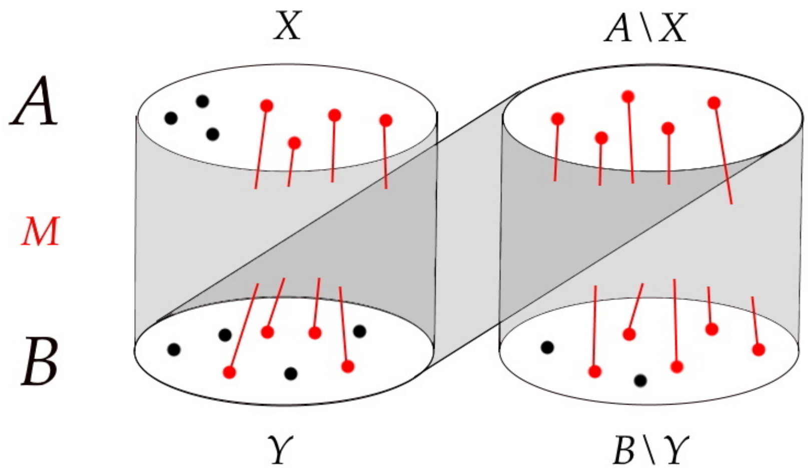


Figure 10.2: A matching in a bipartite graph.

For part (c), suppose that $e = ay \in M$ is a matching edge with $a \in A \setminus X$ and $y \in Y$. As in part (b), there is an M-alternating path $(v_0\, v_1 \ldots v_k)$ with $v_0 \in X_0$ and $v_k = y \in B$. Also as in part (b) by parity and bipartiteness, $v_{k-1}v_k \notin M$ is not a matching edge. Since $ay \in M$ it follows that $(v_0\, v_1 \ldots v_k\, a)$ is an M-alternating path, so that $a \in X$, a contradiction. □

Proof of König's Theorem. Let M be a maximum matching in a graph $G = (V, E)$ with bipartition (A, B). Construct the sets X_0, X, and Y as in Definition 10.11. Since M is a maximum matching, Propositions 10.6 and 10.12(a) imply that every vertex in Y is M-saturated. By Proposition 10.12(c), the edges of M saturating Y provide a bijection between the sets Y and $X \setminus X_0$. By Proposition 10.12(b), the set $C = Y \cup (A \setminus X)$ is a cover of G. Finally,

$$|C| = |Y| + |A \setminus X| = |X \setminus X_0| + |A \setminus X| = |A \setminus X_0| = |M|,$$

so that C is a cover of G of the same size as M. □

Example 10.13. Let $m, d_{\max} \geq 1$ be integers. Let $G = (V, E)$ be a bipartite graph with m edges, in which every vertex has degree at most $d_{\max}$. Then G has a matching with at least $m/d_{\max}$ edges. To see this, consider a maximum matching M of G. By König's Theorem, G has a (minimum) cover C of the same size: $|M| = |C|$. Each vertex is incident with at most

$d_{\max}$ edges, so that C is incident with at most $d_{\max}|C|$ edges. Since C is a cover, we conclude that $m = |E(G)| \leq d_{\max}|C| = d_{\max}|M|$, from which the result follows.

10.2.2 A bipartite matching algorithm.

The proof of Theorem 10.10 can be adapted to provide an algorithm for finding a maximum matching in a bipartite graph. We first describe a "subroutine" which certifies whether or not a given matching is maximum, and then include this in the main algorithm. This description is slightly abstract, requiring you to think abut how to implement it in your favourite computer language. While constructing the sets X and Y of Definition 10.11, Algorithm 10.14 also constructs a (partially defined) "parent function" $\mathrm{pr} : V \to V \cup \{\texttt{null}\}$ for reasons which will become clear.

Algorithm 10.14 (The "XY-algorithm")**.**
Input: a graph $G = (V, E)$ with bipartition (A, B) and matching $M \subseteq E$;
let U be the set of M-unsaturated vertices in A;
for each $x \in U$, let $\mathrm{pr}(x) := \texttt{null}$;
let $X := U$; let $Y := \varnothing$;
let $\texttt{flag} := \texttt{cover}$;
while $U \neq \varnothing$ do:
{ let $U' := \varnothing$;
  for $b \in B \setminus Y$ do:
  { if $N(b) \cap U \neq \varnothing$ then
    { choose any $x \in N(b) \cap U$;
      let $Y := Y \cup \{b\}$; let $\mathrm{pr}(b) := x$;
      if b is M-unsaturated then { let $\texttt{flag} := \texttt{path}$; let $y := b$; break; }
      else { let $ba \in M$; let $U' = U' \cup \{a\}$; let $\mathrm{pr}(a) := b$; }
    }
  }
  let $X := X \cup U$; let $U := U'$;
}
if $\texttt{flag} = \texttt{cover}$ then Output $(\texttt{cover}, Y \cup (A \setminus X))$;
else
{ let $\sigma := y$;

while $\mathrm{pr}(y) \neq \mathtt{null}$ do: $\{$ let $y := \mathrm{pr}(y)$; let $\sigma := \sigma\, y$; $\}$
Output $(\mathtt{path}, (\sigma))$.
$\}$

Informally, Algorithm 10.14 does the following. The first five lines initialize the data. The "do while" loop starting on line six grows the sets X and Y in stages, starting from the initial set of M-unsaturated vertices in A. At each stage, the set U is the set of vertices in X which have just been added, and U' is the set of vertices which will be added to X in the next stage. When $U' = \varnothing$, no new vertices will be added to X, and the loop ends. Inside the loop, if the break command is executed then the algorithm jumps out of the loop immediately. After this loop is the output section: the value of `flag` indicates whether the output is a cover or an M-augmenting path. (The last "do while" loop produces the M-augmenting path from the vertex y and the parent function.)

Example 10.15. We apply Algorithm 10.14 to the graph and matching in Example 10.1. When we add a vertex v to Y or U, the notation $v \to w$ means that $\mathrm{pr}(v) = w$.

U	Y	U'
b, h	$2 \to b, 4 \to b, 5 \to h, 8 \to h$	$a \to 2, d \to 4, e \to 5, g \to 8$
a, d, e, g	$1 \to a, 3 \to a$ break	$c \to 1$

Output: $(\mathtt{path}, (3\, a\, 2\, b))$.

Algorithm 10.16 (Maximum bipartite matching)**.**
Input: a graph $G = (V, E)$ with bipartition (A, B);
let $M := \varnothing$; let $\mathtt{flag} := \mathtt{path}$;
while $\mathtt{flag} = \mathtt{path}$ do:
$\{$ apply Algorithm 10.14 to G and M to get output $(\mathtt{flag}, Q)$;
if $\mathtt{flag} = \mathtt{path}$ then let $M := M \triangle E(Q)$;
$\}$
Output: (M, Q).

Proposition 10.17. *Let $G = (V, E)$ be a graph with bipartition (A, B). Let (M, Q) be the output when Algorithm 10.16 is applied to the input G. Then M is a maximum matching and Q is a minimum cover of G.*

Proof. Exercise 10.9 □

10.3 Hall's Theorem.

Shawn is especially interested in the cases when all the work teams can be assigned jobs. For instance, after a close look at the graph in Figure 10.1 one sees that the five teams {B, D, E, G, H} are among themselves only qualified to do the four jobs $\{2, 4, 5, 8\}$. So there is no way to get all the teams working on jobs they are qualified for. Hall's Theorem 10.18 states that obvious "bottlenecks" like this are the only problems which could arise.

Let $G = (V, E)$ be a graph. For $S \subseteq V$, let the *neighbourhood* of S be

$$N(S) = \{v \in V : \ vw \in E \text{ for some } w \in S\}.$$

Also, a matching M is *S-saturating* when every vertex of S is M-saturated. Thus, a matching is perfect if and only if it is V-saturating.

Theorem 10.18 (Hall's Theorem, 1935)**.** *Let $G = (V, E)$ be a graph with a bipartition (A, B). Then G has an A-saturating matching if and only if for all $S \subseteq A$, $|N(S)| \geq |S|$.*

Proof. First, we see that the condition is necessary. Let M be an A-saturating matching of G. For any subset $S \subseteq A$, since M saturates S, the edges of M incident with vertices in S provide an injective function from S to $N(S)$, so that $|S| \leq |N(S)|$. Conversely, assume that G does not have an A-saturating matching. Let M be a maximum matching of G, and notice that $|M| < |A|$. By König's Theorem 10.10, there is a cover C of G of the same size as M; that is, $|C| = |M|$. Now consider the subset $S = A \smallsetminus C$ of A. Since C is a cover, there are no edges between $A \smallsetminus C$ and $B \smallsetminus C$. Therefore, $N(S) \subseteq B \cap C$.

Now

$$\begin{aligned}|S| = |A \smallsetminus C| = |A| - |A \cap C| = |A| - |C \smallsetminus B|\\ = |A| - (|C| - |B \cap C|) > |B \cap C| \geq N(S).\end{aligned}$$

This completes the proof. □

Corollary 10.19. *Let $G = (V, E)$ be a k-regular bipartite graph. Then the edges of G can be partitioned into k pairwise disjoint perfect matchings.*

Proof. Let G have bipartition (A, B). The case $k = 0$ is trivial, and the case $k = 1$ is when G is itself a matching. We continue by induction on k. By Exercise 5.5, $|A| = |B|$, so that a matching in G is perfect if and only if it is A-saturating. For the induction step, we first find an A-saturating matching in G. To do this we verify Hall's Condition. Consider any subset $S \subseteq A$. Recall the boundary ∂S of S, and notice that $\partial S \subseteq \partial N(S)$. Now

$$k \cdot |S| \ = \ |\partial S| \ \leq \ |\partial N(S)| \ = \ k \cdot |N(S)|.$$

Since $k \geq 1$ it follows that $|S| \leq |N(S)|$. By Hall's Theorem 10.18, there is an A-saturating matching M in G. Now $G \smallsetminus M$ is a $(k-1)$-regular bipartite graph, and we invoke the induction hypothesis to complete the proof. □

Example 10.20. The Petersen graph has exactly six perfect matchings. Any two perfect matchings of the Petersen graph have exactly one edge in common. In particular, it is not possible to partition the edges of the Petersen graph into three pairwise disjoint perfect matchings.

10.4 Exercises.

Exercise 10.1.

(a) Show that a tree has at most one perfect matching.
(b) For which values of $r, s \in \mathbb{N}$ does the grid $P_r \,\square\, P_s$ have a perfect matching?

Exercise 10.2.

(a) For each $n \in \mathbb{N}$, how many perfect matchings are there in K_n?
(b) For each $r, s \in \mathbb{N}$, how many perfect matchings are there in $K_{r,s}$?

Exercise 10.3.

(a) Let G be a graph with a bipartition (A, B) and a Hamilton cycle. Let $a \in A$ and $b \in B$. Show that $G \smallsetminus \{a, b\}$ has a perfect matching.
(b) Show that if two squares of opposite colours are removed from a (standard 8-by-8) chessboard, then the remaining squares can be covered usng 31 dominoes

Exercise 10.4. In the game of Slither (Example 10.3), show that if the graph G has a perfect matching then the first player can always win.

Exercise 10.5. Show that C is a cover in a graph G if and only if $V \smallsetminus C$ induces an edgeless subgraph of G.

Exercise 10.6. Define the "gap" of a graph to be the minimum size of a cover minus the maximum size of a matching. (See Example 10.9.)

(a) For each $k \in \mathbb{N}$, find a connected graph with gap k.
(b) For each $k \in \mathbb{N}$, find a connected graph with gap k and with no bridges.

Exercise 10.7. Let G be a graph with bipartiton (A, B), and let C and C' be covers in G.

(a) Show that $\widehat{C} = ((C \cap C') \cap A) \cup ((C \cup C') \cap B)$ is also a cover.
(b) Show that if C and C' are minimum covers in G then $\widehat{C}$ is also a minimum cover.

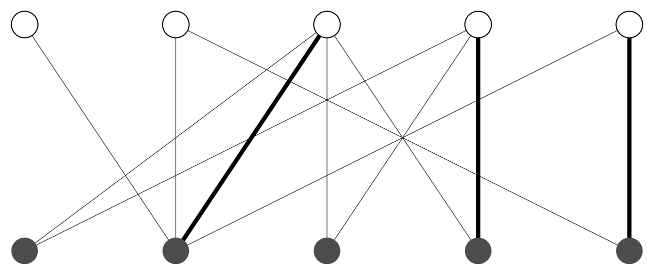


Figure 10.3: Graph for Exercise 10.8(a).

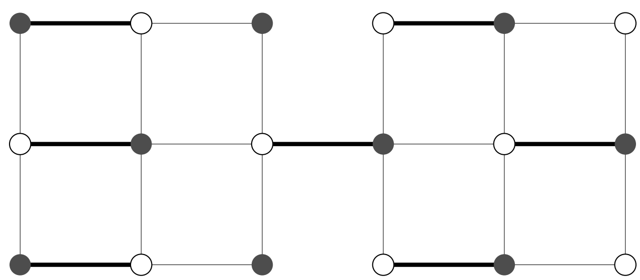


Figure 10.4: Graph for Exercise 10.8(b).

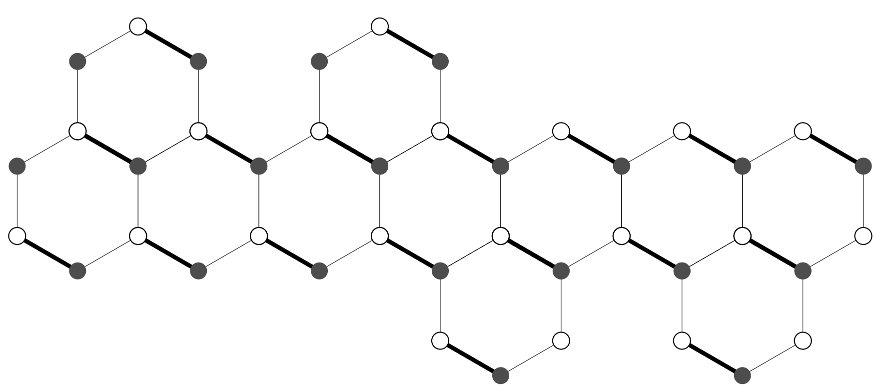


Figure 10.5: Graph for Exercise 10.8(c).

Exercise 10.8. For each part, find a maximum matching and a minimum cover in the pictured graph, by repeated application of the XY-algorithm beginning with the indicated matching.

(a) Figure 10.3.
(b) Figure 10.4.
(c) Figure 10.5.

Exercise 10.9. Prove Proposition 10.17.

Exercise 10.10. Find graph with the following properties: it has 16 edges; it has a bipartition (A, B) with $|A| = |B| = 5$; every vertex has degree at least 2; it does not have a perfect matching. Explain why your example does not have a perfect matching.

Exercise 10.11. Let G be a graph with bipartition (A, B). Assume that there are no vertices of degree zero in A. Also assume that for every edge $ab \in E$, with $a \in A$ and $b \in B$, we have $\deg(a) \geq \deg(b)$. Show that G has an A-saturating matching.

Exercise 10.12. Let $G = (V, E)$ be a graph with bipartition (A, B). Assume that for every proper subset $\varnothing \neq S \subsetneq A$ of A, we have $|N(S)| > |S|$. Let $e \in E$ be any edge. Show that G has an A-saturating matching that contains e.

Exercise 10.13. (This is difficult.) Let $m, k, d_{\min} \geq 1$ be integers. Let G be a graph with m edges, with a bipartition (A, B) such that $|A| = |B| = k$, and such that every vertex has degree at least $d_{\min}$. Assume that $k > d_{\min}$. Show that G has a matching of size at least the minimum of k and $(m - d_{\min}^2)/(k - d_{\min})$.

Exercise 10.14.

(a) Show that König's Theorem 10.10 and Hall's Theorem 10.18 and Corollary 10.19 are also valid more generally for multigraphs in place of graphs.

(b) Deal the 52 cards of a standard deck into an array with 4 rows and 13 columns. Show that it is possible to rearrange the four cards within each column in such a way that each of the four rows contains a card of each rank: 2, 3, ..., Q, K, A.

Exercise 10.15. Let $G = (V, E)$ be a graph and $v \in V$ a vertex. A *splitting of G at v* is any graph H obtained as follows. Let S_1, S_2 be nonempty subsets of $N(v)$ such that $S_1 \cap S_2 = \varnothing$ and $S_1 \cup S_2 = N(v)$. Let z_1, z_2 be new vertices not in V. Then

$$V(H) = (V \cup \{z_1, z_2\}) \smallsetminus \{v\}$$

and

$$E(H) = E(G \smallsetminus v) \cup \{z_1 w : \ w \in S_1\} \cup \{z_2 w : \ w \in S_2\}.$$

(a) Let d be a positive integer such that d divides the degree of every vertex of G. Prove that one can obtain a d-regular graph starting from G by repeatedly splitting vertices.

(b) A graph $G = (V, E)$ with bipartition (A, B) is *(k, ℓ)-biregular* when every vertex in A has degree k and every vertex in B has degree ℓ. Let $m, a, b \geq 1$ be integers, and let G be a (ma, mb)-biregular bipartite graph. Prove that G contains an (a, b)-biregular spanning subgraph.

Exercise 10.16. Prove the claims in Example 10.20.

Exercise 10.17. Fix $k \in \mathbb{N}$. Let $G = (V, E)$ be a graph with $|V| = 2k$ vertices, in which each vertex has degree at least k. Show that G has a perfect matching. [Hint: show that any matching that is not perfect has an augmenting path of length 1 or 3.]